Bridonneau *et al. Applied Network Science* (2024) 9:21
https://doi.org/10.1007/s41109-024-00615-5

Applied Network Science

RESEARCH

Open Access

Asymptotic dynamic graph order evolution analysis

Vincent Bridonneau[1*†], Frédéric Guinand[1†] and Yoann Pigné[1†]

†Vincent Bridonneau, Frédéric Guinand and Yoann Pigné contributed equally to this work.

*Correspondence: vincent.bridonneau@univ-lehavre.fr

[1] RI2C LITIS, Université Le Havre Normandie, 25 Rue Philippe Lebon, 76600 Le Havre, France

Abstract

In this work, we investigate the analysis of generators for dynamic graphs, which are defined as graphs whose topology changes over time. We focus on generated graphs whose order (number of nodes) varies over time. We use a concept called "sustainability" to qualify the long-term evolution of dynamic graphs. A dynamic graph is considered sustainable if its evolution does not result in a static, empty, or periodic graph. To illustrate how the analysis can be conducted, a parameterized and probability-based generator, named D3G3 (Degree-Driven Dynamic Geometric Graphs Generator), has been introduced in a recent work. From this model, we derive multiple scenarios that correspond to three trends in graph order evolution. Our central contribution lies in a mathematical framework that provides an expectation of the order of the graph at time step $t+1$, given its order at time step t. Nevertheless, our analysis underscores the challenge of characterizing the sustainability of dynamic graphs, even when a formal mathematical model for graph order evolution is known.

Keywords: Dynamic graphs, Graph generation, Graph properties, Evolutionary models

Introduction

This work provides an analytical study for generated graphs obtained in the context of dynamic graph generators. A dynamic graph generator can be defined as a computational process that takes input data, such as an initial graph (referred to as a seed graph), and proceeds to generate a sequence of static snapshot graphs. This sequence is generated by applying predefined rules to the previously generated graphs. More precisely, a generator will produce a snapshot graph G_{t+1} at a step $t+1$ considering t generated snapshot graphs $\{G_1, \ldots, G_t\}$ and the seed graph G_0. The output of a dynamic graph generator is therefore a stream of static graphs ordered according to timestamps.

In that context, the present work focuses on the analysis of the evolution of the graph order (number of nodes) of dynamic graphs obtained by a specific generator. Many works have been dedicated to the generation of graphs. Most of them have been designed for a specific purpose (Barabási and Albert 1999; Krioukov et al. 2010; Zuev et al. 2015; Papadopoulos et al. 2012; Muscoloni and Cannistraci 2018; Clementi et al. 2010; Erdős and Rényi 1960; Watts and Strogatz 1998).

Most of the time, the order increases at each time step (growing networks) (Barabási and Albert 1999; Krioukov et al. 2010; Zuev et al. 2015; Papadopoulos et al. 2012; Muscoloni and Cannistraci 2018) or remains the same (Clementi et al. 2010; Erdős and Rényi 1960; Watts and Strogatz 1998). In Tishby et al. (2019) and Tishby et al. (2020) however, the authors study the evolution of graphs subject to three different contraction schemes. Applying their method, the order of the graph is decreasing at each time step. They show that whatever the contraction process, the topology of the graphs converges to the classical Erdös-Renyi graph topology. However, for all these works, the order of generated graphs is known at each step and thus studying the evolution of the order is useless. A more interesting contribution was recently proposed in Budnick et al. (2022). The authors propose to apply an addition-deletion process for driving the evolution of a graph. Starting from an initial graph, equivalent to the seed considered in our model, they propose an evolution method based on two rules. A node-addition rule and a node-deletion rule. At each time step only one rule is applied and its choice is probabilistic, with a probability P_{del} for the node-deletion rule and $1 - P_{add}$ for the node-addtion rule. In their work they study the evolution of the distribution of degrees which is time-dependent. While the proposed generator is interesting for studying many graph properties, in the context of our study, the value of P_{del} determines the property we propose to examine.

The purpose of this work is to address the question in another way. Here the mechanism generating graphs is assumed to be known and the problem is to find properties generated graphs satisfy. As a first study, this work deals with the evolution of graph order when the generator relies on rules enabling both the addition and the deletion of vertices. According to the generative mechanism, it may happen, after some time steps, that generated graphs become empty forever or periodic. A notion called "sustainability" was introduced in Bridonneau et al. (2023a) to highlight this phenomenon. If there exists a time step t such that a generated graph becomes empty or periodic from that moment, then this graph is said to be "non sustainable". Otherwise, if no such time step exists, the graph is said "sustainable".

Definition 1 (*Graph sustainability*) A dynamic graph G is said sustainable if both Condition 1 and Condition 2 are not verified.

$$\begin{aligned}
&\text{Condition 1:} \quad \exists T \in \mathbb{N}, \forall t \geqslant T, G_t = (\emptyset, \emptyset) \\
&\text{Condition 2:} \quad \exists T \in \mathbb{N} \ \text{ and } \ \exists k \in \mathbb{N}^*, \forall t \geqslant T, G_t = G_{t+k}
\end{aligned}$$

To better understand the purpose of this notion, a new version of the Degree Driven Dynamic Geometric Graph Generator (D3G3), introduced in Bridonneau et al. (2023b), is considered. Graphs produced by D3G3 are geometric graphs. A geometric graph is defined by an euclidean space and a threshold d. If the euclidean distance between two distinct vertices is lower that the threshold, they are connected by and edge. For this study, without loss of generality we consider a 2D-unit-torus (i.e., a square $[0; 1[^2$ where the two opposite sides are connected). Each vertex is characterized by a set of coordinates, such that given two vertices u and v it is possible to compute their euclidean distance: $dist(u, v)$. Given V the set of vertices, the set of edges E is defined in the following way: $E = \{(u, v) \in V^2 \mid \text{ dist } (u, v) \leqslant d\}$.

Graphs generated by D3G3 are produced thanks to an evolution process. This mechanism is parameterized by an initial graph (the seed graph) and by two transition rules driving the evolution of the graph between two consecutive time steps. Apart from a random generator, no external decision or additional information is used by this mechanism. Rules are based on node degrees only and rely on a random generator for positioning new nodes in the 2D euclidean space. This leads to the name of the generator: *Degree-Driven Dynamic Geometric Graphs Generator* or D3G3.

Definition 2 (*Degree Driven Dynamic Geometric Graph Generator*) An instance of D3G3 is defined by an initial graph, a set of parameters and two rules:

- $G_0 \neq (\emptyset, \emptyset)$ the seed graph,
- parameters:
  - $d \in]0, \frac{\sqrt{2}}{2}[$
  - S_S a set of non-negative integers
  - S_C a set of non-negative integers
- rules applied on G_t leading to G_{t+1}:
  - if $v \in V_t$, then $v \in V_{t+1}$ iff $\deg(v) \in S_S$ (conservation rule, position of v remains unchanged)
  - if $v \in V_t$ and if $\deg(v) \in S_C$ then a new vertex u is added to V_{t+1} (creation rule, position of u is random in the unit-torus)

The general evolution process is iterative. To compute a new graph at step $i+1$ from the current step i, the generator makes two independent decisions for each node: 1) whether this node from step i will exist in step $i+1$ or should it disappear, and 2) whether the node from step i will generate a new node at $i+1$ (at a random position). These decisions are guided by the node's degree. If the degree value is contained in set S_S, then the node survives to the next step and retains its actual position. If the degree of the node is contained in parameter S_C, then a new node is created. These two processes are independent. Finally, edges between nodes are updated based on proximity.

The order of the graph at each step is not set by any external process or as a parameter of the generator but rather emerges from the application of the rules on consecutive snapshot graphs. The central question is whether, for a given parameter set, the generated graphs are sustainable or not.

However, due to a memory effect (the conserved nodes with their position) between two consecutive steps of the evolution of the graph, we did not found any mathematical approach for the analysis of the sustainability. This led us to propose a relaxed version of the generator.

For the new model considered in this article, position of conserved nodes considered in D3G3 are not kept for the next time step, but are randomly repositioned into the 2D-space so that their new position is independent from one step to the next one. Moreover, this study also restrain the values both sets S_S and S_C may take, so that they are considered to be the same. Therefore parameter sets S_S and S_C will be referred to as

S, a set of non-negative integers. The rules driving the evolution of the generator become as follows: if a node at a given step t has its degree in S then it is conserved and it is at the origin of a new node at step $t+1$. Another consideration is made in the context of this study. Considered values of set S are such that $S = \{sk + r \mid k \in \mathbb{N} + r \in A\}$, for a fixed positive integer s and a set $A \subset [0, s)$. For this very specific settings, we show that generated graphs have one of the three following behavior when their order is big enough: either their order increases exponentially, either it decreases exponentially or it is roughly constant. For each case, we provide criterion about sustainability. We show that proving whether a generated graph is sustainable for this three cases is not obvious and need to consider both small and big snapshot graphs.

Model and concepts

The model discussed here, RD3G3, is a relaxation of the D3G3 model defined in the introduction. It mainly differs on two points. First, only one set of non-negative integers is considered for both the conservation and the creation rules. Second, all conserved vertices are repositionned in the 2D-torus at the following time step. The model is defined as follows:

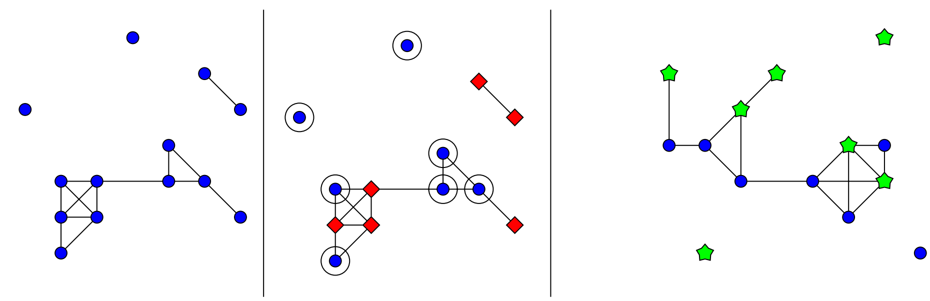
Definition 3 (*Redistributed Degree Driven Dynamic Geometric Graph Generator*) An instance of the model is defined by an initial graph, a set of integer and a rule:

- $G_0 \neq (\emptyset, \emptyset)$ the initial graph,
- parameters:
  - $d \in]0, \frac{\sqrt{2}}{2}[$
  - S a set of non-negative integers
- rules applied on G_t leading to G_{t+1}:
- for all $v \in V_t$ such that $\deg(v) \in S$, $v \in V_{t+1}$ (conservation rule) with a new position and a new vertex is added to V_{t+1} with a random position in the unit-torus (creation rule)

Figure 1 illustrates the application of this rule for one example snapshot G_t to the next one (G_{t+1}). The term “redistributed” here comes from the new treatment of conserved nodes from D3G3. In the D3G3 model, if at a step t a node is conserved at step $t+1$ then its position does not change. Unlike this original version, at every time step t, the conserved nodes at step $t+1$ are uniformly redistributed over the torus so that new graphs are random geometric graphs whose order depends only on the number of conserved nodes. One can then find an estimation function $f_{S,d}$ of graph order at step $t+1$ knowing graph order at step t:

$$\forall n, f_{S,d}(n) = 2n\left(\sum_{k\in S}\binom{n-1}{k}p^k(1-p)^{n-1-k}\right) \tag{1}$$

where p is the probability for two nodes to be connected (for $d \in (0, \frac{1}{2})$, $p = \pi d^2$). Here n refers the order of the graph at step t. For the rest of the article, considered values of S will be restrained. These values are specified in the following section.



(a) G_t (b) Intermediate step. (c) G_{t+1}

Fig. 1 At step *t* the graph is $G_t = (V_t, E_t)$, with $|V_t| = 13$. On this graph, during the intermediate step, rules are applied to every vertex. $S = \{0, 2, 3\}$, each node which degree belongs to *S* is kept and is duplicated (blue circle surrounded by a circle) and all the other nodes are removed (red diamonds). There are 7 nodes that are kept and 6 that are removed. Then new nodes (green stars) are added to the graph and the remaining ones are randomly positioned in the space. This leads to graph G_{t+1}

Asymptotic graph order evolution

This section aims at presenting our work on the RD3G3 model for restrained values on the parameter S. Indeed, this work focuses on sets of the form $S = \{sk + r \mid r \in A, k \in \mathbb{N}\}$ for fixed $s \in \mathbb{Z}^+$ and $A \subset [0, s-1]$. Such sets correspond to non-negative integers that are multiples of a positive integer (s) plus a remainder within set A. The main result of this paper is an equivalent of $f_{S,d}(n)$ for large values of n. This equivalent will also help understanding the behavior of generated graphs with high orders. It will also provide an answer to whether generated graphs are sustainable or not.

Intermediate result

The result of this work relies on properties roots of unity satisfy. As a reminder, a nth root of unity for any positive integer n is defined as follow:

Definition 4 Let n be a positive integer. Then a nth root of unity is a complex number ω such that $\omega^n = 1$.

Such numbers satisfy several properties one may find in Hadlock (2000) at section 2.5. Most important ones for this article are gathered in the following lemma:

Lemma 1 *Let n be a positive integer. Then the following holds*:

- $\omega_n = \exp\left(\frac{2i\pi}{n}\right)$ is a nth root of unity;
- a complex number ω is a nth roots of unity if and only if there exist k such that $\omega = \omega_n^k$;
- if a complex number ω is a nth root of unity, then its modulus satisfies $|\omega| = 1$;
- sum of jth powers of nth root of unity, for any non-negative integer j, are such that:

$$\sum_{k=0}^{n-1} \left(\omega_n^k\right)^j = \begin{cases} n & \text{If } n \text{ divides } j \\ 0 & \text{Otherwise} \end{cases}$$

Such numbers are useful to prove the following result concerning infinite sums:

Lemma 2 *Let $s \in \mathbb{Z}^+, n \in \mathbb{N}$ and $x \in \mathbb{R}$. Let $r \in [0, s-1]$, then we get the following equality*:

$$\sum_{k=0}^{+\infty} \binom{n}{sk+r} x^{sk+r} = \frac{1}{s} \sum_{j=0}^{s-1} \omega_s^{-jr} (1 + \omega_s^j x)^n$$

where $\omega_s = \exp\left(\frac{2i\pi}{s}\right)$ is an sth root of unity.

Proof Let $s \in \mathbb{Z}^+, n \in \mathbb{N}$ and $x \in \mathbb{R}$. Let $r \in [0, s-1]$. Let $\omega_s = \exp\left(\frac{2i\pi}{s}\right)$. The first thing to notice is that the infinite sum on the left side of the equality converges. For any values of k such that $sk + r > n$, the binomial $\binom{n}{sk+r} = 0$. Thus, the infinite sum contains only finitely many non-zero terms. Then, it is sufficient to notice that, according to properties roots of unity satisfy, the following holds:

$$\forall m, \frac{1}{s} \binom{n}{m} x^m \sum_{j=0}^{s-1} \omega_s^{j(m-r)} = \begin{cases} \binom{n}{m} x^m & \text{If there exists} k \text{such that} m = sk + r \\ 0 & \text{Otherwise} \end{cases}$$

From this the following equations hold:

$$\begin{aligned}
\sum_{k=0}^{+\infty} \binom{n}{sk+r} x^{sk+r} &= \frac{1}{s} \sum_{m=0}^{+\infty} \binom{n}{m} x^m \sum_{j=0}^{s-1} (\omega_s^j)^{m-r} \\
&= \frac{1}{s} \sum_{m=0}^{+\infty} \sum_{j=0}^{s-1} \omega_s^{-jr} \binom{n}{m} (\omega_s^j\, x)^m \\
&= \frac{1}{s} \sum_{j=0}^{s-1} \omega_s^{-jr} \left(\sum_{m=0}^{+\infty} \binom{n}{m} (\omega_s^j\, x)^m \right) \\
&= \frac{1}{s} \sum_{j=0}^{s-1} \omega_s^{-jr} \left(\sum_{m=0}^{n} \binom{n}{m} (\omega_s^j\, x)^m \right) \\
&= \frac{1}{s} \sum_{j=0}^{s-1} \omega_s^{-jr} (1 + \omega_s^j\, x)^n
\end{aligned}$$

This ends the proof. □

This lemma on roots of unity helps getting another expression of the function $f_{S,d}$:

Theorem 1 *Let $s \in \mathbb{Z}^+, n \in \mathbb{N}$ and $A \subset [0, s-1]$. Let S as defined above, then*:

$$f_{S,d}(n) = \frac{2}{s} n \left(\sum_{r \in A} \left(\sum_{j=0}^{s-1} \omega_s^{-jr} \left(1 - p + \omega_s^j p\right)^{n-1} \right) \right)$$

Proof Let $s \in \mathbb{Z}^+$, $n \in \mathbb{N}$ and $A \subset [0, s-1]$. Rewriting $f_{S,d}(n)$ lead to the following expression

$$\begin{aligned} f_{S,d}(n) &= 2n \sum_{r \in A} \left(\sum_{k=0}^{+\infty} \binom{n-1}{sk+r} p^{sk+r} (1-p)^{n-1-(sk+r)} \right) \\ &= 2n(1-p)^{n-1} \sum_{r \in A} \left(\sum_{k=0}^{+\infty} \binom{n-1}{sk+r} \left(\frac{p}{1-p} \right)^{sk+r} \right) \end{aligned}$$

Thus, applying result of lemma 2 provides:

$$\begin{aligned} f_{S,d}(n) &= \frac{2}{s} n(1-p)^{n-1} \sum_{r \in A} \left(\sum_{j=0}^{s-1} \omega_s^{-jr} \left(1 + \omega_s^j \frac{p}{1-p} \right)^{n-1} \right) \\ &= \frac{2}{s} n \sum_{r \in A} \left(\sum_{j=0}^{s-1} \omega_s^{-jr} \left(1 - p + \omega_s^j p\right)^{n-1} \right) \end{aligned}$$

This ends the proof. □

This theorem provides an exact formulae for the estimation function $f_{S,d}$. It is important to notice that this re-written formulae involves only finite sums. It is therefore easier to deal with its analysis which is the purpose of the following sub-section.

The equivalent and first interpretation

From result obtained in the last subsection, it is possible to get an equivalent for large values of n for $f_{S,d}$:

Theorem 2 *Let $s \in \mathbb{Z}^+$, $n \in \mathbb{N}$ and $A \subset [0, s-1]$. Let S as defined above, then for large values of n:*

$$f_{S,d}(n) \sim \frac{2|A|}{s} n$$

Proof This comes from Theorem 1 and from properties on complex numbers. More precisely, for each value of $r \in A$ there is exactly one value of $j \in [0, s-1]$ such that $\omega_s^{-jr}(1 - p + \omega_s^j p) = 1$ (for $j = 0$). For all other values of j, $(1 - p + \omega_s^j p) \neq 1$ and have a modulus lower than 1. Therefore, raised to the $n-1$-th power, $(1 - p + \omega_s^j p)^{n-1} \longrightarrow 0$ as n grows to infinity. The rest is computation of limits. □

This result has an interpretation for graphs generated with the model. Indeed, for a given snapshot graph at step t of order n_t, the application of the rule will produce a graph with an expected order $\left(\frac{2|A|}{s}\right) n_t$ at step $t+1$. The evolution of graph order exhibits three different trends depending on whether $\left(\frac{2|A|}{s}\right)$ is lower than, equal to or greater than 1. The next section goes further in the analysis of these three cases. It also highlights the differences between graph order evolution of big and small graphs: interpretation depends on the smallest values of parameter S.

Generated graphs interpretation

This section aims at going further in the interpretation of previously stated results. More precisely, this section highlight three different asymptotic graph order evolution that occur from stated equivalent in . Moreover, interpretation for small graph order is given. This will help knowing whether generated graphs are likely to remain steady or not, depending on the smallest values of the parameter S.

General observations

Before dealing with each case, it is important to understand the meaning of Theorem 2. This theorem states that for any given generated graph having n_t nodes at a step t and assuming n_t is large enough, then, at the next step, n_{t+1} is expected to be close to $\left(\frac{2|A|}{s}\right)n_t$. Therefore, starting with a seed graph of order N large enough would lead, after t steps, to a graph of order

$$n_t \simeq \left(\frac{2|A|}{s}\right)^t N$$

This is why graph order is said to grow exponentially. From this, three cases have to be considered:

- The first case is $\frac{2|A|}{s} < 1$. This means generated graphs order is likely to decrease when it is large.
- The second case is $\frac{2|A|}{s} > 1$. This means generated graphs order is likely to increase when it is large.
- Finally, the third case is $\frac{2|A|}{s} = 1$. This means generated graphs order is likely to remain steady when it is large.

Exponential increasing

The first studied case is when s and A both satisfy $\frac{2|A|}{s} > 1$. For this case, as S is not bounded the order of generated graphs is likely to tend to infinity. Generated graphs are therefore likely to be sustainable. An instance illustrating this case is given Fig. 2.

Exponential decreasing

The second studied case is when s and A both satisfy $\frac{2|A|}{s} < 1$. For this case, graph order of generated graphs is likely to decrease exponentially. An instance illustrating this case is given Fig. 3. However, it is not enough to conclude on the sustainability of generated graphs. Indeed, when graphs become small enough (close to 0), one may consider to take into account the smallest values of set S. This last case is further studied in section .

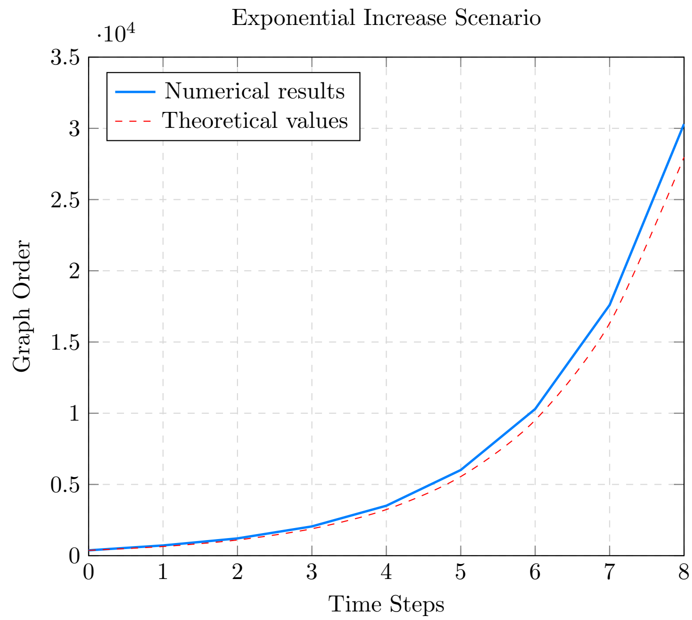


Fig. 2 Scenario of exponential increase. $A = [0, 5], s = 7, n_0 = 375$. The theoretical value is given by the formula $n_t \approx \left(\frac{2|A|}{s}\right)^t n_0$

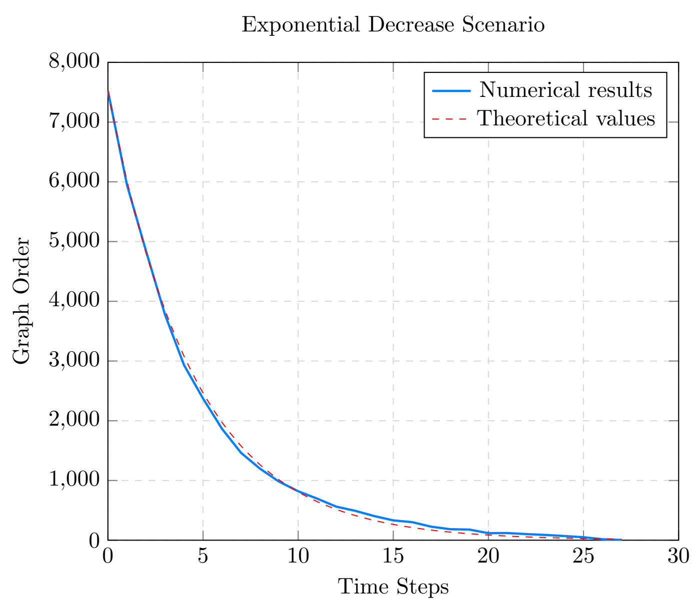


Fig. 3 Scenario of exponential decrease. $A = [4, 5], s = 5, n_0 = 7523$. Theoretical values are given by the formula $n_t \approx \left(\frac{2|A|}{s}\right)^t n_0$

Quasi constant evolution

Two points must be noticed for the last case. First, this case happens if and only if s is even. Indeed, if s is odd, whatever the set A one may choose, the numerator will be even. Second, for a given time step t, application of the rule on a graph which order is n_t will produce a graph which order is expected to be $n_{t+1} = n_t$. It is however necessary to go further as $f_{S,d}$ only provides an expectation. The graph order will indeed change a little. An estimation for this change can be obtained with the standard deviation of a binomial law. Despite all these consideration simulations have been performed. They all show that graph order changes through time with little variations. These simulations are represented in Fig. 4. It is worth noticing graph order is not constant all along the simulation, but rather increasing or decreasing a little bit every time.

A further step to this study is to take into account the standard deviation $\sigma_{S,d}$ associated with graph order evolution. For a given $n_t \in \mathbb{Z}^+$ order of a graph at step t, $\sigma_{S,d}(n_t)$ tells how far away from n_{t+1} is $f_{S,d}(n_t)$, which in this case is roughly n_t. Thus, applying Chebishev's inequality (Feller 1991), for instance, states that for any given real number $k > 0$:

$$\Pr[n_{t+1} \notin [n_t - k\sigma_{S,d}(n_t), n_t + k\sigma_{S,d}(n_t)]] \leqslant \frac{1}{k^2}$$

The computation of $\sigma_{S,d}(n)$ for large enough values of n lead to an equivalent which is the purpose of the following theorem:

Theorem 3 *Let* $s \in \mathbb{Z}^+$, $n \in \mathbb{N}$ *and* $A \subset [0, s-1]$. *Let* S *as defined above, then for large values of* n:

$$\sigma_{S,d}(n) \sim \frac{1}{s}\sqrt{n|A|(s-|A|)}$$

Proof The proof of this theorem relies on the same argument as for theorem 2 and on the definition of the standard deviation of binomial distributions. $\square$

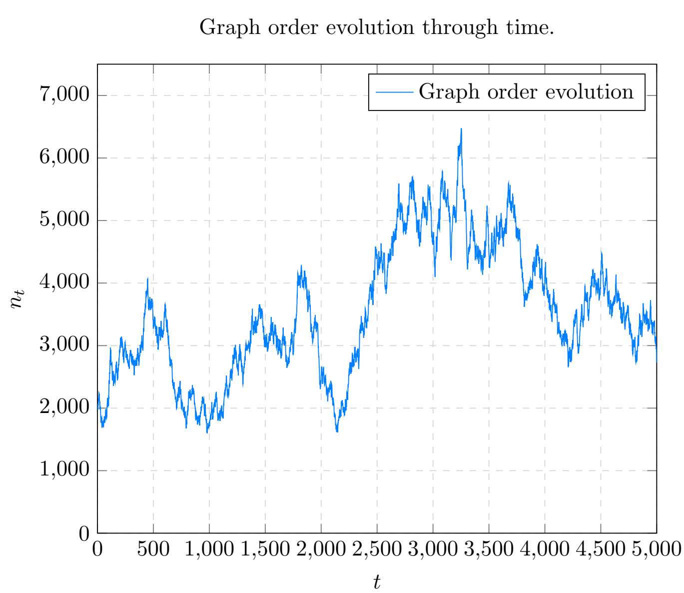


Fig. 4 Simulation performed considering $s = 4$, $A = [0, 1]$ and $d = 0.05$. The number of steps is 5000 and the initial seed graph is a random geometric graph of order 2000

This theorem states that the standard deviation $\sigma_{S,d}(n)$ is proportional to $\sqrt{n}$ for large values of n. This provides better information about the possible values n_{t+1} may have depending on n_t. Indeed, now above stated inequality can by rewritten as follow:

$$\Pr[n_{t+1} \notin [n_t - \frac{k}{2}\sqrt{n_t}, n_t + \frac{k}{2}\sqrt{n_t}]] \leqslant \frac{1}{k^2}$$

Therefore, n_{t+1} and n_t are expected to be roughly the same with a difference expected to be small in comparison to n_t (proportional to $\sqrt{n_t}$). It is however not enough to conclude about the sustainability as nothing prevent the graph order to reach small values. It is necessary to add a focus on small graph order to answer the question about what happens when graphs become small.

Sustainability of small generated graphs

The question of whether a small generated graph is sustainable or not does not depend on the asymptotic variation of the graph order. The answer to this question relies on the smallest values that the parameter S contains.

Indeed, on the one hand, whatever the values of s one may consider, if $A \subset [k, s-1)$ for any $k \geqslant \frac{s}{2}$, then graphs whose order does not exceed k do not have nodes with a degree greater than or equal to k. Therefore such graphs become empty because they do not have any node satisfying the creation rule. A further step is to consider small values of parameter S. For instance, for $d = 0.05$, $s = 16$ and $A = [8, 15]$, the full-lined curve of $f_{S,d}$ represented in Fig. 6 shows that for small values of n, $f_{S,d}(n) < n$. This means that graph order of small graph is expected to decrease between two consecutive steps and graphs are likely to become empty. Therefore generated graphs, for this configuration are likely not be sustainable.

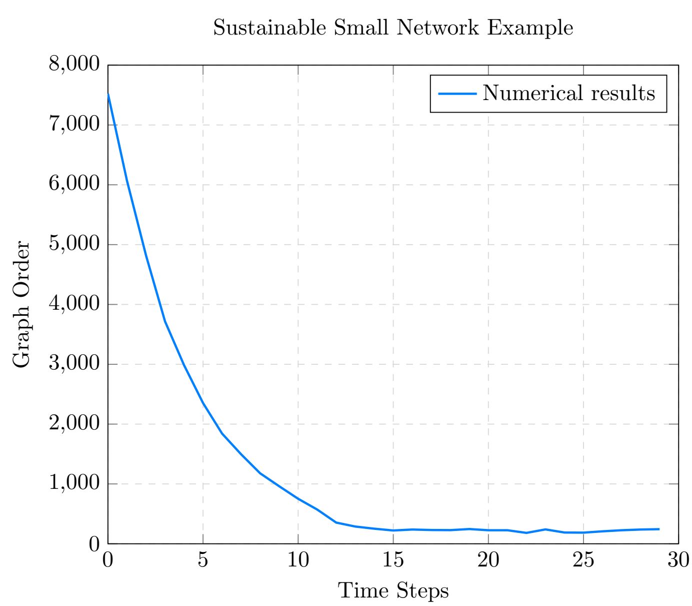


Fig. 5 Example of an instance illustrating the sustainable small network case. $A = [0, 1]$, $s = 5$, $n_0 = 7523$. As A contains small values, the graph is likely to be sustainable since from t to $t + 1$isolated nodes are kept and are at the origin of new nodes. With the current set of parameters, n_t remains close to 210 ± 40

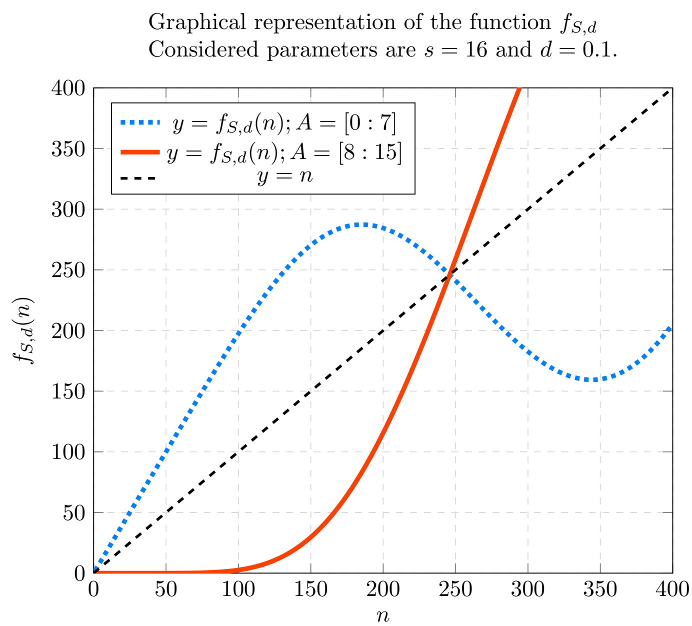


Fig. 6 Theoretical graphical representation of $f_{S,d}$ for value of n from 0 to 400. The blue curve correspond to $A = [0, 7]$ and the red one correspond to $A = [8, 15]$

On the other hand, whatever the values of s one may consider, if $A \subset [0, k+1]$ for any $k < \frac{s}{2}$, then graphs whose order does not exceed k have nodes with a degree lower than or equal to k. Therefore such graphs do not become empty because they have all their nodes satisfying the creation rule. As for the first case, a further step is to consider small values of parameter S. For instance, for $d = 0.05$, $s = 16$ and $A = [0, 7]$, the dotted curve of $f_{S,d}$ represented in Fig. 6 shows that for small values of n, $f_{S,d}(n) \geqslant n$. This means that graph order of small graph is expected to increase between two consecutive steps. Therefore, as soon as graph order does not exceed a certain quantity, generated graphs are likely to conserve few nodes and therefore are likely to be sustainable as illustrated on Fig. 5.

Conclusion

This research work aims to provide insights into the dynamics of dynamic graphs. We propose a metric, called 'sustainability,' to measure the long-term evolution of a dynamic graph. In our context, a graph is considered sustainable if its long-term evolution does not result in an empty graph or a periodic/static graph. To illustrate our approach for analyzing the dynamics, we consider dynamic graphs generated by a modified version of the D3G3 Generator (Bridonneau et al. 2023b). The evolution of such geometric graphs is obtained by the application of two rules on the vertices. Rules are parameterized by an integer set S. Vertices which degree does not belong to S are removed from the graph. The other vertices are kept and duplicated for the next time step. Each vertex is randomly positioned in the environment, thus, vertex degree is stochastic. However, for large graphs, the analysis leads to a mathematical formulation of the evolution of graph order. It has been proved that graph order of generated graphs has three different asymptotic evolutions. Either it is exponentially increasing, exponentially decreasing or quasi constant. For the first case, generated graphs are sustainable with high probability. For the decreasing case, this question is more difficult to answer and sustainability must be considered with respect to sustainability of small generated graphs. Indeed, a decrease in the graph order does not necessarily imply that the graph will disappear. Therefore, it is important to consider the smallest values of the parameter S. Similar considerations apply to graphs exhibiting quasi-constant order with an added focus on standard deviation. However, for the specific configuration studied in this paper, graph order evolution is completely known and yet the sustainable property remains a challenging task. In summary, studying generators that allow both addition and suppression of nodes without a total control on the size of the graph requires an analysis of the sustainability. Our findings illustrate that this property does not yield a straightforward solution, even when the graph order evolution is well-understood and when the model is simple.

Author contributions
All authors wrote the main manuscript. All authors contributed equally in this work. All authors reviewed the manuscript.

Funding
The author(s) disclosed receiving the following financial support for the research, authorship, and/or publication of this article: This project has received funding from the University of Le Havre Normandie.

Declarations

Competing interests
The authors declare no competing interests.

Received: 23 October 2023 Accepted: 15 March 2024
Published online: 10 June 2024

References
Barabási A-L, Albert R (1999) Emergence of scaling in random networks. Science 286(5439):509–512. https://doi.org/10.1126/science.286.5439.509
Bridonneau V, Guinand F, Pigné Y (2023) Dynamic graphs generators analysis: An illustrative case study. In: Doty D, Spirakis PG (eds) 2nd Symposium on algorithmic foundations of dynamic networks, SAND 2023, Pisa, LIPIcs, Schloss Dagstuhl—Leibniz-Zentrum für Informatik, vol 257, pp 8–1819. https://doi.org/10.4230/LIPICS.SAND.2023.8
Bridonneau V, Guinand F, Pigné Y (2023) Dynamic graphs generators analysis: an illustrative case study. Technical report, LITIS, Le Havre Normandie University. https://hal.science/hal-03910386
Budnick B, Biham O, Katzav E (2022) Structure of networks that evolve under a combination of growth and contraction. Phys Rev E 106(4):044305
Clementi AEF, Macci C, Monti A, Pasquale F, Silvestri R (2010) Flooding time of edge-Markovian evolving graphs. SIAM J Discrete Math 24(4):1694–1712. https://doi.org/10.1137/090756053
Erdős P, Rényi A (1960) On the evolution of random graphs. Publ Math Inst Hung Acad Sci 5(1):17–60
Feller W (1991) An introduction to probability theory and its applications. Wiley, New Jersey
Hadlock CR (2000) Field theory and its classical problems. American Mathematical Society, Rhodes Island
Krioukov D, Papadopoulos F, Kitsak M, Vahdat A, Boguñá M (2010) Hyperbolic geometry of complex networks. Phys Rev E 82(3):036106. https://doi.org/10.1103/PhysRevE.82.036106
Muscoloni A, Cannistraci CV (2018) A nonuniform popularity-similarity optimization (nPSO) model to efficiently generate realistic complex networks with communities. New J Phys 20(5):052002. https://doi.org/10.1088/1367-2630/aac06f
Papadopoulos F, Kitsak M, Serrano MA, Boguna M, Krioukov D (2012) Popularity versus similarity in growing networks. Nature 489(7417):537–540. https://doi.org/10.1038/nature11459
Tishby I, Biham O, Katzav E (2019) Convergence towards an erdős-rényi graph structure in network contraction processes. Phys Rev E 100(3):032314
Tishby I, Biham O, Katzav E (2020) Analysis of the convergence of the degree distribution of contracting random networks towards a Poisson distribution using the relative entropy. Phys Rev E 101(6):062308
Watts DJ, Strogatz SH (1998) Collective dynamics of "small-world" networks. Nature 393(6684):440–442. https://doi.org/10.1038/30918
Zuev K, Boguñá M, Bianconi G, Krioukov D (2015) Emergence of soft communities from geometric preferential attachment. Sci Rep 5(1):9421. https://doi.org/10.1038/srep09421

Publisher's Note