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Persistent homology based Bottleneck distance in hypergraph products

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Abstract

In this paper, we extended the technique of measuring similarity between topological spaces using bottle neck distance between persistence diagrams to hypergraph networks. Finding a relationship between the bottleneck distance of the Cartesian product of topological spaces and the bottleneck distance of individual spaces, we are trying to ease the comparative study of the Cartesian product of topological spaces. The Cartesian product and the strong product of weighted hypergraphs are defined, and the relationship between the bottleneck distance between hypergraph products and the bottleneck distance between individual hypergraphs is determined. For this, clique complex filtration and the Vietoris–Rips filtration in unweighted and weighted hypergraphs are defined and used.

Keywords: Persistent homology, Bottleneck distance, Cartesian product of hypergraphs, Strong product of hypergraphs

Introduction

One of the challenges that data science has faced in recent years is obtaining valuable information from complex data sets. In the field of topological data analysis, there has been substantial advancement in dealing with the challenge of analysing the structure of such data. As such, a variety of efficient techniques are available in this rapidly developing field to help with finding significant topological structures in data. By applying homology, an effective mathematical technique, one can categorise objects based on their topological properties. When identifying holes in objects with different dimensions, homology plays an essential role in helping to classify those objects. Due to its efficacy in storing data and its computing efficiency, the simplicial method to homology has been gaining importance. The most advanced technique for identifying topological features is persistent homology, which takes provided data and converts it into simplicial complexes to give an accurate picture of the structure of space at various spatial scales. Through persistent homology, a set of homology classes that are persistent across broad spatial resolutions is identified and these classes represent significant aspects of the underlying space, offering valuable insights into its structure. Two efficient tools for displaying persistent Betti numbers (Pears 1975), which give information about the lifetime and persistence of topological features, are persistence diagrams

(Pears 1975; Edelsbrunner and Harer 2022) and barcodes (Carlsson et al. 2004; Ghrist 2008). Patrizio Frozini and his coworkers first proposed the idea of persistent homology in 1990 (Frosini 1990). They also proposed the use of size functions to determine 0-dimensional persistent homology by identifying connected components. Based on this study, Vanessa Robbins explored sample space homology by characterizing persistent homology groups as a collection of homomorphisms that were generated by inclusion (Robins 1999). Later, Edelsbrunner, Letscher, and Zomorodian developed a definition of persistent homology based on simplicial complexes that are widely recognized (Edelsbrunner and Zomorodian 2002).

Graph theoretic techniques are frequently used to analyze complex systems as they are typically depicted as a collection of entities, or vertices with binary relationships. Graph models, while simple and relatively universal, are limited to representing pairwise relationships between entities. However, actual occurrences such as computer networks, in which dynamic connections are established through packets exchanged over time between computers, or co-authorship networks, where relationships are formed by articles written by two or more authors, can be intricate with multi-way connections, dependencies among more than two variables, or characteristics of collections comprising more than two objects. Thus introduced the powerful tool hypergraph, a broad natural representation that can implicitly capture multi-way relationships and it is introduced in 1973 by Berge. Applications for hypergraphs can be found in many different domains, including knowledge representation (Zhenyong et al. 2018), social network analysis (Zhu et al. 2018; Li et al. 2013; Zlatić et al. 2009), data mining (Gunopulos et al. 1997; Alam et al. 2021), and bioinformatics (Tian et al. 2009; Mithani et al. 2009). To some extent, the concept of 2 section of a hypergraph (Bretto 2013) is useful and easy while comparing hypergraphs. In the work of Aktas et al. (2023) they have defined persistent homology for hypergraph through filtrations using simplicial complex closure which is simply 2 section of hypergraph. So this filtration is same as the clique filtration that we have defined in this paper. Consequently, when comparing hypergraphs topologically, 2 section and its filtration are effectively useful. To find the 0-dimensional and 1-dimensional persistence barcodes for the given hypergraphs, we may use the filtration 2 section of the hypergraphs. If the barcodes are not similar, we could conclude that the given hypergraphs are not isomorphic. As so, we can compare hypergraphs to some extent through barcodes, filtering, and persistence diagram comparison (Bandyopadhyay et al. 2020).

Using the idea of a categorical product, Gakhar and Perea (2019)have presented a method to look at the filtration of the Cartesian product of topological spaces. Additionally, they have proposed a technique for ascertaining the barcodes of the Cartesian product of topological spaces, which offer significant insights into the topological properties. A powerful mathematical tool for studying topological features of a data set is a persistence diagram (Pears 1975; Edelsbrunner and Harer 2022), which is used in topological data analysis. It provides a clear description of the underlying geometric and topological structure of the data by capturing the genesis and termination points of topological features like holes and connected components. since persistence diagrams enable us to measure the similarity of topological spaces to some extent, it is possible to extend this idea to measure the similarity of Cartesian products of topological spaces. Furthermore, comparing persistence diagrams is sufficient to compare topological spaces. Among the distances assigned for it is the bottleneck distance. The relationship between the bottleneck distance of the Cartesian product, and individual spaces is discussed in this paper. Since hypergraphs are topological structures, we can also apply this notion to them.

The topological properties and interconnection of certain networks are studied recently (Siddiqui et al. 2016; Dongchuan et al. 2006; Hong et al. 2020). Analogously, studies has been carried out with hypergraphs (Flamm et al. 2015). Our main concern is the hypergraph product. Although the literature has discussed an amazing variety of different products of hypergraphs, most of hypergraph products can be thought of as an extension of one of the four standard graph products. Here, the focus is on the Cartesian product and strong product of simple, undirected and connected hypergraphs. It is hard to compare hypergraphs using common methods, but with the aid of different filtering techniques, graphs can be compared using bottle neck distance (Edelsbrunner and Harer 2022). These filtering methods helps in reducing the complexity of complex hypergraphs to simpler representations, such as products of smaller hypergraphs. This is what in fact we do here because, when complex hypergraph networks can be expressed as a product of these smaller hypergraphs, it is necessary to find out whether there is a relationship between the distance between complex hypergraph networks and the distance between smaller hypergraphs. We have also thought about weighted hypergraphs and their filtration in addition to unweighted hypergraphs. On weighted hypergraphs, we define the Cartesian product (Cooper and Dutle 2012) and the strong product (Hellmuth et al. 2012). Additionally, adding cliques (simplex) (Aktas et al. 2019) and using weight as a parameter for clique filtration for comparing persistence diagrams with 0 and 1 dimensions.

The background information including preliminary definitions, results, theorems, and notations, is provided in "Preliminaries" section of the paper. By introducing the main concepts and frameworks needed for the analysis, it lays the foundation for the sections that follow. The paper presents significant results about the comparison of persistence diagrams in "New Results" section. The relationship between the bottleneck distances of the individual spaces and the bottle distances of persistence diagrams of Cartesian products of topological spaces is specifically studied. Furthermore, the study analyzes the relation between the bottleneck distances of persistence diagrams of Cartesian product and strong product of hypergraphs, with a special focus on smaller hypergraphs. Knowing the Cartesian product and the strong product operations on these hypergraph structures enables us to look into these relationships for both weighted and unweighted hypergraphs.

For fundamental definitions, notations, and terminologies associated with Homology, Persistent homology, and homology of product spaces we can refer to Hatcher (2002), Munkres (2018), Edelsbrunner and Harer (2022), Carlsson et al. (2004), Gakhar and Perea (2019), Wallace (1957). Further, for graph and hypergraph theoretical concepts we refer West (2001), Zhang and Chartrand (2006), Hammack et al. (2011), Bretto (2013), Cooper and Dutle (2012), Hellmuth et al. (2012).

Preliminaries

It is obviously challenging to determine the homology of a topological object, regardless of how complex it may be. A simplicial complex can be used as an alternate way to approximate the topological object. Homology can be computed using this method, called simplicial homology. The following definitions provide a detailed explanation of the requirements that must be followed while defining simplicial complexes. It's also important to understand the basic concepts of chains, cycles, and boundaries before moving into the idea of simplicial homology, an effective tool for locating voids or gaps inside a system. The foundation for dealing with simplicial complexes and recognizing their homology is this fundamental idea.

Definition 2.1 (Wallace 1957; Edelsbrunner and Harer 2022) A simplex is a generalization of the notion of a triangle or tetrahedron to arbitrary dimensions. Specifically, a s simplex is a s -dimensional polytope which is the convex hull of $s+1$ affinely independent points.

Definition 2.2 (Wallace 1957; Edelsbrunner and Harer 2022) A simplicial complex S is a collection of simplices such that

1. If S contains a simplex s_1, then S also contains every face of s_1
2. If two simplices in S intersect, then their intersection is a face of each of them.

Definition 2.3 (Wallace 1957; Edelsbrunner and Harer 2022) Let S be a simplicial complex and p a dimension. A p-chain is a formal sum of p-simplices in S with integer coefficients. The standard notation for this is $c_p = \sum z_i \sigma_i$, where the σ_i are the p-simplices and the z_i are the coefficients. The set of all p-chains form a group C_p under addition.

Definition 2.4 (Wallace 1957; Edelsbrunner and Harer 2022) The boundary of a p-simplex is the sum of its $(p-1)$-dimensional faces. If $\sigma = [u_0, u_1, \ldots, u_p]$ for the simplex spanned by the listed vertices, then its boundary is $\partial_p \sigma = \sum_{j=0}^{p} (-1)^j [u_0, \ldots, \hat{u}_j, \ldots, u_p]$, where ∂_p is called boundary operator and the hat indicates that u_j is omitted.

Definition 2.5 (Wallace 1957; Edelsbrunner and Harer 2022) A p-cycle c is a p-chain with empty boundary, $\partial_p c = 0$. A p-boundary b is a p-chain that is the boundary of a $(p+1)$-chain, $b = \partial_{p+1} d$ with $d \in C_{p+1}$. And the set of all p-cycles and p-boundaries will form subgroups of chain group C_p.

Definition 2.6 (Wallace 1957; Edelsbrunner and Harer 2022) Let C_p be a chain group whose elements are the p chains and $\partial_p : C_p \to C_{p-1}$ maps each p-chain to the sum of the $(p-1)$dimensional faces of its p cells which is a $(p-1)$ chain.

Writing the groups and maps in sequence, we get the chain complex,

$$\ldots \xrightarrow{\partial_{p+2}} C_{p+1} \xrightarrow{\partial_{p+1}} C_p \xrightarrow{\partial_p} C_{p-1} \xrightarrow{\partial_{p-1}} \ldots$$

Then the nth homology group is defined as

$$H_n = Ker(\partial_n)/Im(\partial_{n+1}).$$

Homology may not yield pertinent insights when dealing with point cloud data. To address this limitation, an alternative approach involves constructing a series of simplicial complexes using a method known as filtration. Using certain distances or criteria, filtration involves constructing simplicial complexes from the given points in a systematic way. This procedure creates a series of complexes that, at various scales, capture the data's underlying topological properties. Indeed, a few voids or holes may arise and then disappear during the construction of simplicial complexes and homology computation. We may say that an essential part of the dataset is the homological features' persistence, which shows how long these voids persist on various levels.

Definition 2.7 (Edelsbrunner and Harer 2022) Consider a real valued function $g' : T \to R$ is defined on a topological space T. Let $T_a = g'^{-1}(-\infty, a]$ denote the sublevel set for the function value a. So we have inclusions:

$$T_a \subseteq T_b \text{ for } a \leq b$$

This inclusion induces a map in the homology groups. So, if $i : T_a \to T_b$ denotes the inclusion map, we have induced map

$$\mathrm{f} = \mathrm{i}_* : \mathrm{H_p(T_a)} \to \mathrm{H_p(T_b)}$$

Consider the sequence a sequence of distinct values $a_1 < a_2 < \ldots$ corresponding to which we have the sequence of homomorphisms induced by inclusions.

$$0 \to H_p(T_{a_1}) \to H_p(T_{a_2}) \to H_p(T_{a_3}) \to \ldots \to H_p(T_{a_n}) \to H_p(T)$$

Then the homomorphism

$$\mathrm{f^{ij}} : \mathrm{H_p(T_{a_i})} \to \mathrm{H_p(T_{a_j})}$$

for all p and $1 \leq i \leq j \leq n$ takes the homology classes of the sublevel set T_{a_i} to those of the sublevel sets of T_{a_j}.

The pth persistent homology groups are the images of the homomorphisms:

$$H_p^{ij} = im\mathrm{f_p^{ij}} \text{ for } 1 \leq \mathrm{i} \leq \mathrm{j}.$$

Topological persistence may be introduced with the observation that a nested sequence of topological spaces

$$T_0 \subseteq T_1 \subseteq T_2 \ldots \subseteq T_n$$

gives a sequence of vector spaces and linear maps

$$H_p(T_0) \to H_p(T_1) \to \ldots \to H_p(T_n)$$

upon computing homology with coefficients in a field $\mathbb{F}$. In general, a diagram of vector spaces and linear maps $V_0 \to V_1 \to \ldots \to V_n$ is called a persistent module indexed by $0, 1, 2, \ldots, n$. We can write persistent homology module:

$$M_h = H_p(K_1) \oplus H_P(K_2) \oplus \ldots \oplus H_p(K_n)$$

Module M_h decomposes in to a direct sum of interval modules M_{hj}^p, each of which corresponds to a bar in the barcode (bcd_n).

Definition 2.8 (Bubenik and Scott 2014) A category, $\mathcal{C}$, consists of a class of objects, $\mathcal{C}_0$, and for each pair of objects $\mathcal{X}_1, \mathcal{X}_2 \in \mathcal{C}_0$, a set of morphisms, $\mathcal{C}(\mathcal{X}_1, \mathcal{X}_2)$. We often write $f : \mathcal{X}_1 \to \mathcal{X}_2$ if $f \in \mathcal{C}(\mathcal{X}_1, \mathcal{X}_2)$. For every triple $\mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3 \in \mathcal{C}_0$, there is a set mapping,

$$\mathcal{C}(\mathcal{X}_2, \mathcal{X}_3) \times \mathcal{C}(\mathcal{X}_1, \mathcal{X}_2) \to \mathcal{C}(\mathcal{X}_1, \mathcal{X}_3), \;\; (g,f) \to gf,$$

called composition. Composition must be associative, in the sense that $(hg)f = h(gf)$. Finally, for all $\mathcal{W} \in \mathcal{C}$, there is an identity morphism, $\mathcal{I}d_{\mathcal{W}} : \mathcal{W} \to \mathcal{W}$, that satisfies $\mathcal{I}d_{\mathcal{W}} f = f$ and $g\mathcal{I}d_{\mathcal{W}} = g$ for all $f : \mathcal{Z} \to \mathcal{W}$ and all $g : \mathcal{W} \to \mathcal{Y}$. The identity morphism is unique.

Theorem 2.9 (Gakhar and Perea 2019) *Let P_c be the poset category of a separable(with respect to the order topology) totally ordered set. Let $K_1, K_2 \in S^{P_c}$ be P_c-indexed diagrams of spaces, and assume that $H_i(K_1; \mathbb{F})$ and $H_i(K_2; \mathbb{F})$ are pointwise finite for each $0 \le i, j \le n$ where H_i is the ith persistence homology group. Then $H_n(K_1 \times K_2; \mathbb{F})$ is pointwise finite, and its barcode satisfies*:

$$bcd_n(K_1 \times K_2; \mathbb{F}) = \bigcup_{i+j=n} \left\{ I \cap J \mid I \in \mathrm{bcd}_i(K_1), J \in \mathrm{bcd}_j(K_2) \right\}$$

where the union on the right is of multisets.

Corollary 2.10 (Gakhar and Perea 2019) *Let $K_1, \ldots, K_m \in S^{P_c}$. Assume that for each $1 \le j \le m$ and $0 \le n_j \le n$, then $H_n(K_1 \times K_2 \times \ldots \times K_m)$ is point wise finite, and its barcode satisfies*:

$$bcd_n(K_1 \times K_2 \times \ldots \times K_m) = \{I_1 \cap I_2 \cap \ldots I_k | I_j \in \mathrm{bcd}_{n_j}(K_j), \sum_{j=1}^{m} n_j = n\}$$

where the union on the right is of multi sets.

Definition 2.11 (Aktas et al. 2019) The clique complex $\mathscr{C}\ \mathscr{L}(G)$ of an undirected graph $G = (W, F)$ is a simplicial complex where vertices of G are its vertices and each k-clique, i.e. the complex sub graphs with k vertices, in G corresponds to a $(k-1)$-simplex in $\mathscr{C}\ \mathscr{L}(G)$.

Definition 2.12 (Edelsbrunner and Harer 2022) Let $\mathscr{P}$ and $\mathscr{Q}$ be two persistence diagrams. The bottleneck distance between $\mathscr{P}$ and $\mathscr{Q}$ is defined as

$$\hat{d}_B(\mathscr{P}, \mathscr{Q}) = \inf_{\alpha} \sup_{x \in \mathscr{P}} \|x - \alpha(x)\|_\infty$$

where α ranges over all matchings from $\mathscr{P}$ to $\mathscr{Q}$ and $\|p - q\|_\infty = \max(|p_1 - q_1|, |p_2 - q_2|)$ for $p = (p_1, p_2), q = (q_1, q_2) \in \mathbb{R}^2$ with $\|\infty - \infty\| = 0$.

Hypergraphs are a natural generalization of undirected graphs in which "edges" may consist of more than 2 vertices (Berge 1984). More precisely, a (finite) hypergraph $\mathrm{H} = (\mathrm{V_H}, \mathrm{E_H})$ consists of a (finite) set V_{H} and a collection E_{H} of non-empty subsets of

V_{H}. The elements of V_{H} are called vertices and the elements of E_{H} are called hyperedges, or simply edges of the hypergraph (Hellmuth et al. 2012).

A hypergraph $\mathrm{H} = (\mathrm{V_H}, \mathrm{E_H})$ is simple if no edge is contained in any other edge and $|e| \geq 2$ for all $e \in E_{\mathrm{H}}$. A hypergraph $\mathrm{H} = (\mathrm{V_H}, \mathrm{E_H})$ is called connected, if any two vertices are joined by a path (Hellmuth et al. 2012).

For two hypergraphs $\mathrm{H}_1 = (\mathrm{V}_{\mathrm{H}_1}, \mathrm{E}_{\mathrm{H}_1})$ and $\mathrm{H}_2 = (\mathrm{V}_{\mathrm{H}_2}, \mathrm{E}_{\mathrm{H}_2})$ a homomorphism from H_1 into H_2 is a mapping $\Phi : V_{\mathrm{H}_1} \rightarrow V_{\mathrm{H}_2}$ such that $\Phi(e) = \{\Phi(v_1), \ldots, \Phi(v_r)\}$ is an edge in H_2, if $e = \{v_1, \ldots, v_r\}$ is an edge in H_1. A homomorphism $V_{\mathrm{H}_1} \rightarrow V_{\mathrm{H}_2}$ such that $\Phi(e) = \{\Phi(v_1), \ldots, \Phi(v_r)\}$ is an edge in H_2, if $e = \{v_1, \ldots, v_r\}$ is an edge in H_1. A homomorphism Φ that is bijective is called an isomorphism if holds $\Phi(e) \in E_{\mathrm{H}2}$ if and only if $e \in E_{\mathrm{H}_1}$. We say, H_1 and H_2 are isomorphic, in symbols $\mathrm{H}_1 \cong \mathrm{H}_2$ if there exists an isomorphism between them (Hellmuth et al. 2012).

Definition 2.13 (Hellmuth et al. 2014) The 2-section $[\mathrm{H}]_2$ of a hypergraph $\mathrm{H} = (\mathrm{V_H}, \mathrm{E_H})$ is the graph (V_{H}, E') with $E' = \{\{x, y\} \subseteq V_{\mathrm{H}} \mid x \neq y, \exists\, e \in E_{\mathrm{H}} : \{x, y\} \subseteq e\}$, that is, two vertices are adjacent in $[\mathrm{H}]_2$ if they belong to the same hyperedge in H. Thus, every hyperedge of H is a clique in $[\mathrm{H}]_2$.

Definition 2.14 (Cooper and Dutle 2012) The Cartesian product $\mathrm{H}_1 \square \mathrm{H}_2$ of two hypergraphs H_1 and H_2 has vertex set $V_{\mathrm{H}_1 \square \mathrm{H}_2} = V_{\mathrm{H}_1} \times V_{\mathrm{H}_2}$ and the edge set $E_{\mathrm{H}_1 \square \mathrm{H}_2} = \{x \times f : x \in V_{\mathrm{H}_1}, f \in E_{\mathrm{H}_2}\} \bigcup \{e \times y : e \in E_{\mathrm{H}_1}, y \in V_{\mathrm{H}_2}\}$.

Example 2.15

Let H_1 *be a hyper graph with vertices* $V_{\mathrm{H}_1} = \{1, 2, 3\}$ *and a hyperedge* $e_{\mathrm{H}_1} = \{1, 2, 3\}$ *and* H_2 *be a hyper graph with vertices* $V_{\mathrm{H}_2} = \{a, b, c\}$ *and a hyperedge* $e_{\mathrm{H}_2} = \{a, b, c\}$ *(Fig. 1).*

Definition 2.16 (Hellmuth et al. 2012) Let $\mathrm{H}_1 = (V_{\mathrm{H}_1}, E_{\mathrm{H}_1})$ and $\mathrm{H}_2 = (V_{\mathrm{H}_2}, E_{\mathrm{H}_2})$ be two hypergraphs. Then the strong product $\mathrm{H}_1 \boxtimes \mathrm{H}_2$ of H_1 and H_2 is defined as $\mathrm{H}_1 \boxtimes \mathrm{H}_2 = (V_{\mathrm{H}_1 \boxtimes \mathrm{H}_2}, E_{\mathrm{H}_1 \boxtimes \mathrm{H}_2})$ where

$$V_{\mathrm{H}_1 \boxtimes \mathrm{H}_2} = V_{\mathrm{H}_1} \times V_{\mathrm{H}_2}$$

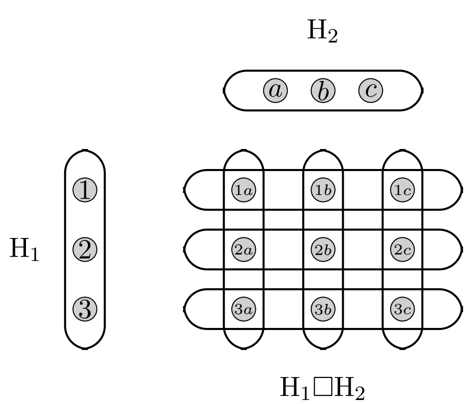

Fig. 1 Cartesian product of hypergraphs

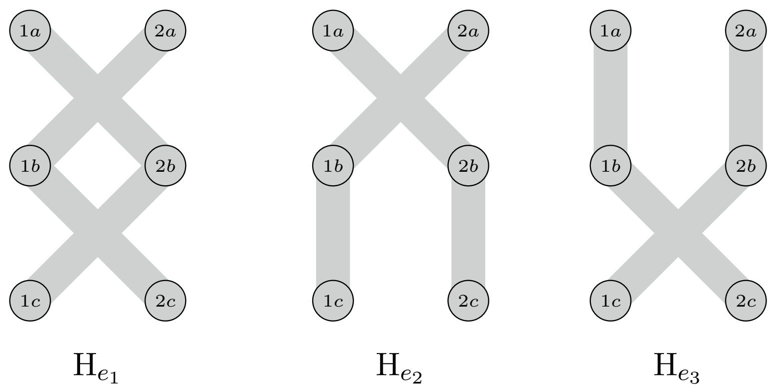

Fig. 2 Non-Cartesian edges of strong product of hypergraphs

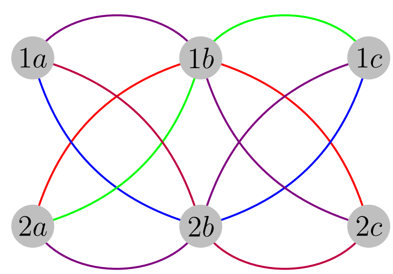

Fig. 3 non-Cartesian edges of strong product of hypergraphs

and a subset $e = \{(v_1, v_1'), (v_2, v_2'), \ldots, (v_n, v_n')\}$ of $V_{H_1} \times V_{H_2}$ is an edge in edge set $E_{H_1 \boxtimes H_2}$ of $H_1 \boxtimes H_2$ if,

1. $\{v_1, v_2, \ldots, v_n\}$ is an edge of H_1 and $v_1' = v_2' = \cdots = v_n' \in V_{H_2}$, or
2. $\{v_1', v_2', \ldots, v_n'\}$ is an edge of H_2 and $v_1 = v_2 = \cdots = v_n \in V_{H_1}$, or
3. $\{v_1, v_2, \ldots, v_n\}$ is an edge of H_1 and there is an edge $f \in E_{H_2}$ such that $\{v_1', v_2', \ldots, v_n'\}$ is a multi set of elements of f, and $f \subseteq \{v_1', v_2', \ldots, v_n'\}$, or
4. $\{v_1', v_2', \ldots, v_n'\}$ is an edge of H_2 and there is an edge $f \in E_{H_1}$ such that $\{v_1, v_2, \ldots, v_n\}$ is a multi set of elements of f, and $f \subseteq \{v_1, v_2, \ldots, v_n\}$.

Example 2.17
Let H_1 be a hyper graph with vertices $V_{H_1} = \{a, b, c\}$ and a hyperedge $e_{H_1} = \{a, b, c\}$ and H_1' be a hyper graph with vertices $V_{H_2} = \{1, 2\}$ and a hyperedge $e_{H_2} = \{1, 2\}$. Then the non Cartesian edges of strong product of these two hypergraphs will be the union of hyperedges of $H_{e_1}, H_{e_2}, H_{e_3}$ (Fig. 2).

Using a single graph, we can represent the non-Cartesian edges of the strong productH$_1 \boxtimes H_2$. With the help of this method, the combined edges can be visualized in their entirety, giving a clear picture of their relationship in relation to H_1 and H_2 (Fig. 3).

New results

Here even if we are doing the direct filtration of hyper graphs by considering parameter as size of each edge will be topologically similar to the filtration of two section. So we can represent $bcd_k([\mathrm{H}_1 \Box \mathrm{H}_2]_2)$ as simply $bcd_k(\mathcal{H}_1 \Box \mathcal{H}_2)$. Also for any filtrations $\mathcal{X}$ and $\mathcal{Y}$ of topological spaces X and Y we will have to define new term $d_{\min}(\mathscr{P}^k, \mathscr{Q}^k)$ with kth persistence diagrams $\mathscr{P}^k$ and $\mathscr{Q}^k$ as

$$d_{\min}(\mathscr{P}^k, \mathscr{Q}^k) = \min_{i,j=1,2,\ldots,k} \{|a_{i_l} - c_{j_t}|, |b_{i_l} - d_{j_t}|,\ l,t = 1,2,\ldots,n\}$$

where,

$$bcd_i(\mathcal{X}) = \{(a_{i_1}, b_{i_1}), (a_{i_2}, b_{i_2}), \ldots, (a_{i_n}, b_{i_n})\}$$

and

$$bcd_i(\mathcal{Y}) = \{(c_{i_1}, d_{i_1}), (c_{i_2}, d_{i_2}), \ldots, (c_{i_n}, d_{i_n})\}$$

with $i + j = k$.

Theorem 3.1 *Let $\mathcal{X}, \mathcal{Y}, \mathcal{X}'$, and $\mathcal{Y}'$ be filtrations of topological spaces X, Y, X', and Y' respectively with kth persistence diagrams $\mathscr{P}^k, \mathscr{Q}^k, \mathscr{P}'^k$, and $\mathscr{Q}'^k$. If $(\mathscr{P} \times \mathscr{Q})^k$ and $(\mathscr{P}' \times \mathscr{Q}')^k$ are kth persistence diagrams of filtrations $\mathcal{X} \times \mathcal{Y}$ and $\mathcal{X}' \times \mathcal{Y}'$ of Cartesian products $X \times Y$ and $X' \times Y'$ respectively. Then,*

$$d_B((\mathscr{P} \times \mathscr{Q})^k, (\mathscr{P}' \times \mathscr{Q}')^k) \geq \min\{d_{\min}(\mathscr{P}^k, \mathscr{P}'^k), d_{\min}(\mathscr{P}^k, \mathscr{Q}'^k), d_{\min}(\mathscr{P}'^k, \mathscr{Q}^k), d_{\min}(\mathscr{P}'^k, \mathscr{Q}'^k)\}$$

Proof For $i = 0, 1, 2, \ldots, k$, let

$$bcd_i(\mathcal{X}) = \{(a_{i_1}, b_{i_1}), (a_{i_2}, b_{i_2}), \ldots, (a_{i_n}, b_{i_n})\}$$
$$bcd_i(\mathcal{Y}) = \{(c_{i_1}, d_{i_1}), (c_{i_2}, d_{i_2}), \ldots, (c_{i_n}, d_{i_n})\}$$
$$bcd_i(\mathcal{X}') = \{(a'_{i_1}, b'_{i_1}), (a'_{i_2}, b'_{i_2}), \ldots, (a'_{i_n}, b'_{i_n})\}$$

and

$$bcd_i(\mathcal{Y}') = \{(c'_{i_1}, d'_{i_1}), (c'_{i_2}, d'_{i_2}), \ldots, (c'_{i_n}, d'_{i_n})\}.$$

We have, for all $i + j = k$,

$$d_{\min}(\mathscr{P}^k, \mathscr{P}'^k) = \min_{i,j=1,2,\ldots,k} \{|a_{i_l} - a'_{j_t}|, |b_{i_l} - b'_{j_t}|,\ l,t = 1,2,\ldots,n\}$$
$$d_{\min}(\mathscr{P}^k, \mathscr{Q}'^k) = \min_{i,j=1,2,\ldots,k} \{|a_{i_l} - c'_{j_t}|, |b_{i_l} - d'_{j_t}|,\ l,t = 1,2,\ldots,n\}$$
$$d_{\min}(\mathscr{P}'^k, \mathscr{Q}^k) = \min_{i,j=1,2,\ldots,k} \{|a'_{j_t} - c_{i_q}|, |b'_{j_t} - d_{i_q}|,\ q,t = 1,2,\ldots,n\}$$

and

$$d_{\min}(\mathcal{Q}^{k}, \mathcal{Q}'^{k}) = \min_{i,j=1,2,\ldots,k}\{|c_{j_t} - c'_{i_q}|, |d_{j_t} - d'_{i_q}|,\ q, t = 1, 2, \ldots, n\}.$$

Let $(\mathcal{P} \times \mathcal{Q})^k$ be the k dimensional persistence diagrams of $\mathcal{X} \times \mathcal{Y}$ and $(\mathcal{P}' \times \mathcal{Q}')^k$ be the k dimensional persistence diagrams of $\mathcal{X}' \times \mathcal{Y}'$. For any $(a, b) \in bcd_k(\mathcal{X} \times \mathcal{Y})$ and $(c, d) \in bcd_k(\mathcal{X}' \times \mathcal{Y}')$,

$$(a, b) = (a_{i_l}, b_{i_l}) \cap (c_{w_s}, d_{w_s}) \text{ for some } i, w \in \{0, 1, \ldots, k\},\ i + w = k,\ l, s \in \{1, 2, \ldots, n\}$$

and

$$(c, d) = (a'_{o_r}, b'_{o_r}) \cap (c'_{p_t}, d'_{p_t}) \text{ for some } i, w \in \{0, 1, \ldots, k\},\ o + p = k,\ r, t \in \{1, 2, \ldots, n\}.$$

Here

$$a = \max\{a_{i_l}, c_{w_s}\},\ b = \min\{b_{i_l}, d_{w_s}\}$$

and

$$c = \max\{a'_{o_r}, c'_{p_t}\},\ d = \min\{b'_{o_r}, d'_{p_t}\}.$$

Then,

$$d_B((\mathcal{P} \times \mathcal{Q})^k, (\mathcal{P}' \times \mathcal{Q}')^k) \geq \min\{d_{\min}(\mathcal{P}^{k}, \mathcal{P}'^{k}), d_{\min}(\mathcal{P}^{k}, \mathcal{Q}'^{k}),$$
$$d_{\min}(\mathcal{P}'^{k}, \mathcal{Q}^{k}), d_{\min}(\mathcal{P}'^{k}, \mathcal{Q}'^{k})\}$$

□

While doing Clique filtration we are considering parameters from 0 to ∞. Hence the minimum value of each interval in zero dimensional barcode will be zero. In Bottleneck distance we are taking maximum values. So in the case of zero dimensional persistence diagrams we have to consider the following terms,

$$d_{B\min}(\mathcal{P}^{k}, \mathcal{P}'^{k}) = \min_{i,j=1,2,\ldots,k}\{|b_{i_l} - b'_{j_t}|,\ l, t = 1, 2, \ldots, n\}$$
$$d_{B\min}(\mathcal{P}^{k}, \mathcal{Q}'^{k}) = \min_{i,j=1,2,\ldots,k}\{|b_{i_l} - d'_{j_t}|,\ l, t = 1, 2, \ldots, n\}$$
$$d_{B\min}(\mathcal{P}'^{k}, \mathcal{Q}^{k}) = \min_{i,j=1,2,\ldots,k}\{|b'_{j_t} - d_{i_q}|,\ q, t = 1, 2, \ldots, n\}$$

and

$$d_{B\min}(\mathcal{Q}^{k}, \mathcal{Q}'^{k}) = \min_{i,j=1,2,\ldots,k}\{|b'_{j_t} - d'_{i_q}|,\ q, t = 1, 2, \ldots, n\},\ \text{for } k = 0, 1$$

Corollary 3.2 *Let $\mathcal{X}_1, \mathcal{X}_2, \ldots, \mathcal{X}_n$ and $\mathcal{Y}_1, \mathcal{Y}_2, \ldots, \mathcal{Y}_n$ are the filtrations of topological spaces $X_1, X_2, \ldots, X_n$ with kth persistence diagrams $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n$ and $Y_1, Y_2, \ldots, Y_n$ with kth persistence diagrams $\mathcal{Q}_1, \mathcal{Q}_2, \ldots, \mathcal{Q}_n$ respectively. If $(\mathcal{P}_1 \times \mathcal{P}_2 \times \cdots \times \mathcal{P}_n)^k$ be the kth persistence diagram of $\mathcal{X}_1 \times \mathcal{X}_2 \times \ldots \times \mathcal{X}_n$ and $(\mathcal{Q}_1 \times \mathcal{Q}_2 \times \cdots \times \mathcal{Q}_n)^k$ be the kth persistence diagram of $\mathcal{Y}_1 \times \mathcal{Y}_2 \times \ldots \times \mathcal{Y}_n$, then*

$$d_B((\mathscr{P}_1 \times \mathscr{P}_2 \times \cdots \times \mathscr{P}_n)^k, (\mathscr{Q}_1 \times \mathscr{Q}_2 \times \cdots \times \mathscr{Q}_n)^k) \geq \min_{g,h=1,2,\ldots,n}\{d_{\min}({\mathscr{P}_g}^k, {\mathscr{Q}_h}^k)\}.$$

Proof For $i_j = 1, 2, \ldots, k, j = 1, 2, \ldots, n$, let

$$\begin{aligned}
bcd_{i_1}(\mathcal{X}_1) &= \{(a_{i_{1_1}}, b_{i_{1_1}}), (a_{i_{1_2}}, b_{i_{1_2}}), \ldots, (a_{i_{1_m}}, b_{i_{1_m}})\} \\
bcd_{i_2}(\mathcal{X}_2) &= \{(a_{i_{2_1}}, b_{i_{2_1}}), (a_{i_{2_2}}, b_{i_{2_2}}), \ldots, (a_{i_{2_m}}, b_{i_{2_m}})\} \\
&\vdots \\
bcd_{i_n}(\mathcal{X}_n) &= \{(a_{i_{n_1}}, b_{i_{n_1}}), (a_{i_{n_2}}, b_{i_{n_2}}), \ldots, (a_{i_{n_m}}, b_{i_{n_m}})\}
\end{aligned}$$

and

$$\begin{aligned}
bcd_i(\mathcal{Y}_1) &= \{(c_{i_{1_1}}, d_{i_{1_1}}), (c_{i_{1_2}}, d_{i_{1_2}}), \ldots, (c_{i_{1_m}}, d_{i_{1_m}})\} \\
bcd_i(\mathcal{Y}_2) &= \{(c_{i_{2_1}}, d_{i_{2_1}}), (c_{i_{2_2}}, d_{i_{2_2}}), \ldots, (c_{i_{2_m}}, d_{i_{2_m}})\} \\
&\vdots \\
bcd_i(\mathcal{Y}_n) &= \{(c_{i_{n_1}}, d_{i_{n_1}}), (c_{i_{n_2}}, d_{i_{n_2}}), \ldots, (c_{i_{n_m}}, d_{i_{n_m}})\}.
\end{aligned}$$

Any element (s, t) in $bcd_k(\mathcal{X}_1 \times \mathcal{X}_2 \times \ldots \times \mathcal{X}_n)$ will be of the form

$$(s, t) = (a_{i_{1_{t_1}}}, b_{i_{1_{t_1}}}) \cap (a_{i_{2_{t_2}}}, b_{i_{2_{t_2}}}) \cap \cdots \cap (a_{i_{n_{t_n}}}, b_{i_{n_{t_n}}})$$

for some $t_1, t_2, \ldots, t_n = 1, 2, \ldots, m,$ and $i_1 + i_2 + \cdots + i_n = k$.

Here

$$s = \max\{a_{i_{1_{t_1}}}, a_{i_{2_{t_2}}}, \ldots, a_{i_{n_{t_n}}}\}$$

and

$$t = \min\{b_{i_{1_{t_1}}}, b_{i_{2_{t_2}}}, \ldots, b_{i_{n_{t_n}}}\}$$

also any element (s', t') in $bcd_k(\mathcal{Y}_1 \times \mathcal{Y}_2 \times \ldots \times \mathcal{Y}_n)$ will be of the form

$$(s', t') = (c_{i_{1_{t_1}}}, d_{i_{1_{t_1}}}) \cap (c_{i_{2_{t_2}}}, d_{i_{2_{t_2}}}) \cap \cdots \cap (c_{i_{n_{t_n}}}, d_{i_{n_{t_n}}})$$

for some $t_1, t_2, \ldots, t_n = 1, 2, \ldots, m,$ and $i_1 + i_2 + \cdots + i_n = k$.

Here

$$s' = \max\{c_{i_{1_{t_1}}}, c_{i_{2_{t_2}}}, \ldots, c_{i_{n_{t_n}}}\}$$

and

$$t' = \min\{d_{i_{1_{t_1}}}, d_{i_{2_{t_2}}}, \ldots, d_{i_{n_{t_n}}}\}.$$

Then,

$$d_B((\mathscr{P}_1 \times \mathscr{P}_2 \times \cdots \times \mathscr{P}_n)^k, (\mathscr{Q}_1 \times \mathscr{Q}_2 \times \cdots \times \mathscr{Q}_n)^k) \geq \min_{g,h=1,2,\ldots,n}\{d_{\min}({\mathscr{P}_g}^k, {\mathscr{Q}_h}^k)\}.$$

□

Definition 3.3 Let H be a simple unweighted connected hyper graph and $[\mathrm{H}]_2$ be the 2-section of H. Now for any $n \in \mathbb{Z}$, the clique complex of $[\mathrm{H}]_2$ is denoted as $\mathscr{C}\ \mathscr{L}([\mathrm{H}]_2)$ and the filtration is defined as

$$\mathscr{C}\ \mathscr{L}_0([\mathrm{H}]_2) \rightarrow \mathscr{C}\ \mathscr{L}_1([\mathrm{H}]_2) \rightarrow \cdot \rightarrow \mathscr{C}\ \mathscr{L}_{\mathrm{n}}([\mathrm{H}]_2) = [\mathrm{H}]_2$$

where, $\mathscr{C}\ \mathscr{L}_i([\mathrm{H}]_2) = \sum_{\mathrm{j=1}}^{\mathrm{i}} \mathrm{k_j}$, k_j is the jth skeleton of clique complex.The set of all simplices with dimension less than or equal to j is called the ith complex in this situation. Filtration will be from 0 to ∞. In other words, adding vertices at the parameter of filtration, $\delta = 0$, edges at $\delta = 1$ and so on.

Definition 3.4 Let H_1 and H_2 two simple unweighted connected hyper graphs with Cartesian product $\mathrm{H}_1 \Box \mathrm{H}_2$. Also we have $[\mathrm{H}_1 \Box \mathrm{H}_2]_2$ be the 2-section of of $\mathrm{H}_1 \Box \mathrm{H}_2$. Now for any $n \in \mathbb{Z}$, the clique complex of $[\mathrm{H}_1 \Box \mathrm{H}_2]_2$ is defined as

$$\mathscr{C}\ \mathscr{L}_0([\mathrm{H}_1 \Box \mathrm{H}_2]_2) \rightarrow \mathscr{C}\ \mathscr{L}_1([\mathrm{H}_1 \Box \mathrm{H}_2]_2) \rightarrow \cdot \rightarrow \mathscr{C}\ \mathscr{L}_{\mathrm{n}}([\mathrm{H}_1 \Box \mathrm{H}_2]_2) = [\mathrm{H}_1 \Box \mathrm{H}_2]_2$$

where, $\mathscr{C}\ \mathscr{L}_i([\mathrm{H}_1 \Box \mathrm{H}_2]_2) = \sum_{\mathrm{j=1}}^{\mathrm{i}} \mathrm{K_j}$, K_j is the jth skeleton of clique complex. Here the set of all simplices of dimension less than or equal to j is the ith complex. Filtration starting from 0 to ∞. That is adding vertices at the parameter, $\eta = 0$, edges at $\eta = 1$ and so on.

Definition 3.5 Let $\mathrm{H}_1, \mathrm{H}_2, \ldots, \mathrm{H}_\mathrm{n}$ are simple unweighted connected hyper graphs with Cartesian product $\mathrm{H}_1 \Box \mathrm{H}_2 \Box \cdots \Box \mathrm{H}_\mathrm{n}$. Also we have $[\mathrm{H}_1 \Box \mathrm{H}_2 \Box \cdots \Box \mathrm{H}_\mathrm{n}]_2$ be the 2-section of of $\mathrm{H}_1 \Box \mathrm{H}_2 \Box \cdots \Box \mathrm{H}_\mathrm{n}$. Now for any $n \in \mathbb{Z}$, the clique complex of $[\mathrm{H}_1 \Box \mathrm{H}_2 \Box \cdots \Box \mathrm{H}_\mathrm{n}]_2$ is defined as

$$\mathscr{C}\ \mathscr{L}_0([\mathrm{H}_1 \Box \mathrm{H}_2 \Box \cdots \Box \mathrm{H}_\mathrm{n}]_2) \rightarrow \mathscr{C}\ \mathscr{L}_1([\mathrm{H}_1 \Box \mathrm{H}_2 \Box \cdots \Box \mathrm{H}_\mathrm{n}]_2) \rightarrow \cdots \rightarrow \mathscr{C}\ \mathscr{L}_\mathrm{n}([\mathrm{H}_1 \Box \mathrm{H}_2 \Box \cdots \Box \mathrm{H}_\mathrm{n}]_2)$$

with $\mathscr{C}\ \mathscr{L}_n([\mathrm{H}_1 \Box \mathrm{H}_2 \Box \cdots \Box \mathrm{H}_\mathrm{n}]_2) = [\mathrm{H}_1 \Box \mathrm{H}_2 \Box \cdots \Box \mathrm{H}_\mathrm{n}]_2$ where, $\mathscr{C}\ \mathscr{L}_i([\mathrm{H}_1 \Box \mathrm{H}_2 \Box \cdots \Box \mathrm{H}_\mathrm{n}]_2) = \sum_{\mathrm{j=1}}^{\mathrm{i}} \mathrm{K_{nj}}$, K_{nj} is the jth skeleton of clique complex. Here the ith complex is the set of all simplices o dimension less than or equal to j. Filtration is from 0 to ∞. That is adding vertices at the parameter, $\eta = 0$, edges at $\eta = 1$ and so on.

Theorem 3.6 *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ are the filtrations of two simple unweighted connected hyper graphs* H_1 *and* H_2 *with Cartesian product* $\mathrm{H}_1 \Box \mathrm{H}_2$. *If $\mathcal{H}_1 \Box \mathcal{H}_2$ be the filtration of* $\mathrm{H}_1 \Box \mathrm{H}_2$, *then*

$$bcd_0(\mathcal{H}_1 \Box \mathcal{H}_2) = \{I \cap J \mid I \in bcd_0(\mathcal{H}_1), J \in bcd_0(\mathcal{H}_2)\}.$$

Proof $\mathcal{H}_1$ and $\mathcal{H}_2$ are the filtrations of two simple unweighted connected hyper graphs H_1 and H_2 with Cartesian product $\mathrm{H}_1 \Box \mathrm{H}_2$ and $[\mathrm{H}_1]_2, [\mathrm{H}_2]_2$ are 2 sections of H_1 and H_2 respectively. Let

$$bcd_0(\mathcal{H}_1) = \{(0, a_1), (0, a_2) \ldots (0, a_n) = (0, \infty)\} \text{ with } a_1 \leq a_2 \leq \cdots \leq a_n$$

and

$$bcd_0(\mathcal{H}_2) = \{(0,b_1),(0,b_2)\ldots,(0,b_n) = (0,\infty)\} \text{ with} b_1 \le b_2 \le \cdots \le b_n.$$

Suppose $a_1 \le a_i, b_j$ for all $i = 2,3,\ldots n$, $j = 1,2\ldots n$. While doing filtration, if $\eta = a_1$, there will be n number of $(0,a_1)$ intervals in $bd_0(\mathcal{H}_1 \square \mathcal{H}_2)$. We can represent it as

$$(0,a_1) = (0,a_1) \cap (0,b_j) \text{ for all } j = 1,2,\ldots,n.$$

Now if $a_2 \le a_k, b_j$ for all $k = 3,4,\ldots,n, j = 1,2,\ldots,n$, there will be again n number of $(0,a_2)$ intervals in $bcd_0(\mathcal{H}_1 \square \mathcal{H}_2)$. It can be written as

$$(0,a_2) = (0,a_2) \cap (0,b_j) \text{ for all } j = 1,2,\ldots,n.$$

If $b_1 \le a_m, b_p$ for all $m = 3,4,\ldots,n$ and $p = 2,3,\ldots,n$, then there we have to consider only $(n-2)$ components of H_1 and so that $(n-2)$ times the interval $(0,b_1)$ will be there in $bcd_0(\mathcal{H}_1 \square \mathcal{H}_2)$. It can be represented as

$$(0,b_1) = (0,a_m) \cap (0,b_1) \text{ for all } m = 3,4,\ldots,n.$$

Continuing like this we will get

$$bcd_0(\mathcal{H}_1 \square \mathcal{H}_2) = \{I \cap J \mid I \in bcd_0(\mathcal{H}_1), J \in bcd_0(\mathcal{H}_2)\}.$$

□

Corollary 3.7 *Let $\mathcal{H}_1, \mathcal{H}_2, \ldots, \mathcal{H}_n$ are the filtrations of n simple unweighted connected hyper graphs* $\mathrm{H}_1, \mathrm{H}_2, \ldots, \mathrm{H}_\mathrm{n}$ *with Cartesian product* $\mathrm{H}_1 \square \mathrm{H}_2 \square \cdots \square \mathrm{H}_\mathrm{n}$. *If $\mathcal{H}_1 \square \mathcal{H}_2 \square \cdots \square \mathcal{H}_n$ be the filtration of* $\mathrm{H}_1 \square \mathrm{H}_2 \square \cdots \square \mathrm{H}_\mathrm{n}$, *then*

$$bcd_0(\mathcal{H}_1 \square \mathcal{H}_2 \square \cdots \square \mathcal{H}_n) = \{I_1 \cap I_2 \cap \cdots \cap I_n \mid I_1 \in bcd_0(\mathcal{H}_1), I_2 \in bcd_0(\mathcal{H}_2), \ldots, I_1 \in bcd_0(\mathcal{H}_n)\}.$$

Proof Let

$$\begin{aligned} bcd_0(\mathcal{H}_1) &= \{(0,a_{1_1}),(0,a_{1_2}),\ldots,(0,a_{1_n})\} \\ bcd_0(\mathcal{H}_2) &= \{(0,a_{2_1}),(0,a_{2_2}),\ldots,(0,a_{2_n})\} \\ &\vdots \\ bcd_0(\mathcal{H}_n) &= \{(0,a_{n_1}),(0,a_{n_2}),\ldots,(0,a_{n_n})\}. \end{aligned}$$

Without loss of generality, assume that $a_{1_1} \le a_{i_j}$ for all $i = 1,2,\ldots,n, j = 1,2,\ldots,n$. Then during filtration there will be n^{n-1} number of intervals of the form $(0,a_{1_1})$ and we can represent it as

$$(0,a_{1_1}) = (0,a_{1_1}) \cup (0,a_{2_j}) \cup \cdots \cup (0,a_{n_j}) \quad \text{for all } \; j = 1,2,\ldots,n.$$

If $a_{1_2} \le a_{i_j}$ for all $i = 2,3,\ldots,n. j = 1,2,\ldots,n$ and $a_{1_2} \le a_{j_j}$ for all $j = 1,2,\ldots,n$, then there will be again n^{n-1} intervals of the form $(0,a_{1_2})$ and we can represent it as

$$(0,a_{1_2}) = (0,a_{1_2}) \cup (0,a_{2_j}) \cup \cdots \cup (0,a_{n_j}) \quad \text{for all } \; j = 1,2,\ldots,n.$$

If $a_{k_l} \leq a_{i_j}$ for all $i = 1, 2, \ldots, n, j = 1, 2, \ldots, n$ with $a_{1_1} \leq a_{1_2} \leq a_{k_l}$ for some $k = 1, 2, \ldots, n, l = 1, 2, \ldots, n$, then there will be $n^{n-1} - 2$ possibilities. We can represent this interval as

$$(0, a_{k_l}) = (0, a_{1_j}) \cup (0, a_{2_j}) \cup \cdots \cup (0, a_{k_l}) \cup \cdots \cup (0, a_{n_j}) \quad \text{for all } j = 1, 2, \ldots, n, a_{1_j} \neq a_{1_1}, a_{1_2}.$$

Continuing like this we will get

$$bcd_0(\mathcal{H}_1 \Box \mathcal{H}_2 \Box \cdots \Box \mathcal{H}_n) = \{I_1 \cap I_2 \cap \cdots \cap I_n \mid I_1 \in bcd_0(\mathcal{H}_1), I_2 \in bcd_0(\mathcal{H}_2), \ldots, I_1 \in bcd_0(\mathcal{H}_n)\}.$$

□

Theorem 3.8 *Let* $\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3$, and $\mathcal{H}_4$ *be the filtrations of hyper graphs* $\mathrm{H}_1, \mathrm{H}_2, \mathrm{H}_3$ and H_4 *respectively with* 0*-dimensional persistence diagrams* $\mathscr{P}^0_{\mathrm{H}_1}, \mathscr{P}^0_{\mathrm{H}_2}, \mathscr{P}^0_{\mathrm{H}_3}$ and $\mathscr{P}^0_{\mathrm{H}_4}$. *If* $\mathcal{H}_1 \Box \mathcal{H}_2, \mathcal{H}_3 \Box \mathcal{H}_4$ *be the filtrations of Cartesian products* $\mathrm{H}_1 \Box \mathrm{H}_2$ and $\mathrm{H}_3 \Box \mathrm{H}_4$ *with* 0*-dimensional persistence diagrams* $\mathscr{P}^0_{\mathrm{H}_1 \Box \mathrm{H}_2}, \mathscr{P}^0_{\mathrm{H}_3 \Box \mathrm{H}_4}$, *then*

$$d_B(\mathscr{P}^0_{\mathrm{H}_1 \Box \mathrm{H}_2}, \mathscr{P}^0_{\mathrm{H}_3 \Box \mathrm{H}_4}) \geq \min_{v=3,4} \{d_{B\min}(\mathscr{P}^0_{\mathrm{H}_1}, \mathscr{P}^0_{\mathrm{H}_v}), d_{B\min}(\mathscr{P}^0_{\mathrm{H}_2}, \mathscr{P}^0_{\mathrm{H}_v})\}.$$

Proof We have

$$bcd_0(\mathcal{H}_1 \Box \mathcal{H}_2) = \{I \cap J \mid I \in bcd_0(\mathcal{H}_1), J \in bcd_0(\mathcal{H}_2)\}$$

and

$$bcd_0(\mathcal{H}_3 \Box \mathcal{H}_4) = \{I \cap J \mid I \in bcd_0(\mathcal{H}_3), J \in bcd_0(\mathcal{H}_4)\}.$$

Which means the 0-dimensional of Cartesian product of two hyper graphs is same as the Cartesian product of topological spaces. So by the above theorem,

$$d_B(\mathscr{P}^0_{\mathrm{H}_1 \Box \mathrm{H}_2}, \mathscr{P}^0_{\mathrm{H}_3 \Box \mathrm{H}_4}) \geq \min_{v=3,4} \{d_{B\min}(\mathscr{P}^0_{\mathrm{H}_1}, \mathscr{P}^0_{\mathrm{H}_v}), d_{B\min}(\mathscr{P}^0_{\mathrm{H}_2}, \mathscr{P}^0_{\mathrm{H}_v})\}.$$

□

The minimum value intervals in one dimensional persistence diagrams will be always greater than zero since at zero there won't be any one dimensional hole. So it is enough to consider $d_{\min}$.

Theorem 3.9 *Let* $\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3$, and $\mathcal{H}_4$ *be the filtrations of hyper graphs* $\mathrm{H}_1, \mathrm{H}_2, \mathrm{H}_3$ and H_4 *respectively with* 1*-dimensional persistence diagrams* $\mathscr{P}^1_{\mathrm{H}_1}, \mathscr{P}^1_{\mathrm{H}_2}, \mathscr{P}^1_{\mathrm{H}_3}$ and $\mathscr{P}^1_{\mathrm{H}_4}$. *If* $\mathcal{H}_1 \Box \mathcal{H}_2, \mathcal{H}_3 \Box \mathcal{H}_4$ *be the filtrations of Cartesian products* $\mathrm{H}_1 \Box \mathrm{H}_2$ and $\mathrm{H}_3 \Box \mathrm{H}_4$ *with* 1*-dimensional persistence diagrams* $\mathscr{P}^1_{\mathrm{H}_1 \Box \mathrm{H}_2}, \mathscr{P}^1_{\mathrm{H}_3 \Box \mathrm{H}_4}$, *then*

$$d_B(\mathscr{P}^1_{\mathrm{H}_1 \Box \mathrm{H}_2}, \mathscr{P}^1_{\mathrm{H}_3 \Box \mathrm{H}_4}) \geq \min_{u=1,2,\ v=3,4} \{d_{B\min}(\mathscr{P}^0_{\mathrm{H}_u}, \mathscr{P}^0_{\mathrm{H}_v}), d_{\min}(\mathscr{P}^1_{\mathrm{H}_u}, \mathscr{P}^1_{\mathrm{H}_v})\}.$$

Proof Consider the two sections of hyper graphs $\mathrm{H}_1, \mathrm{H}_2, \mathrm{H}_3$ and H_4. Let $\{v_1, v_2, \ldots, v_n\}$ be the vertices of $[\mathrm{H}_1]_2$ and $\{v'_1, v'_2, \ldots, v'_n\}$ be the vertices of $[\mathrm{H}_2]_2$. Then for any $v'_j \in [\mathrm{H}_2]_2$ the $[\mathrm{H}_1]_2$ fibre in $[\mathrm{H}_1 \Box \mathrm{H}_2]_2$ is defined as

$$[\mathrm{H}_1]_2 \mathrm{v}'_\mathrm{j} = \{(\mathrm{v_i}, \mathrm{v}'_\mathrm{j}) | \mathrm{v_i} \in [\mathrm{H}_1]_2, \mathrm{i} = 1, 2, \ldots, \mathrm{n}\},\ \mathrm{j} = 1, 2, \ldots, \mathrm{n}$$

and for any $v_j \in [\mathrm{H}_1]_2$ the $[\mathrm{H}_2]_2$ fibre in n copies of $[\mathrm{H}_1]_2$ is defined as

$$v_j[\mathrm{H}_2]_2 = \{(\mathrm{v_j}, \mathrm{v}'_\mathrm{i}) | \mathrm{v_i} \in [\mathrm{H}_1]_2, \mathrm{i} = 1, 2, \ldots, \mathrm{n}\},\ \mathrm{j} = 1, 2, \ldots, \mathrm{n}.$$

Clearly $[\mathrm{H}_1]_2$ fibre is isomorphic to $[\mathrm{H}_1]_2$ and $[\mathrm{H}_2]_2$ fibre is isomorphic to $[\mathrm{H}_2]_2$. So there will be n copies of $[\mathrm{H}_1]_2$ and n copies of $[\mathrm{H}_2]_2$ in $\mathrm{H}_1 \Box \mathrm{H}_2$. Which means all the 1-dimensional holes in H_1 and H_2 will be there in $\mathrm{H}_1 \Box \mathrm{H}_2$ n times. Some other loops also will be there and which will be of the form $(t, \infty), 0 \le t \le \infty$ with $t = \max\{t_1, t_2\}$ where $(0, t_1) \in bcd_0(\mathcal{H}_1)$ and $(0, t_2) \in bcd_0(\mathcal{H}_2)$. If we are considering all these cases, we can conclude that

$$d_B(\mathscr{P}^1_{\mathrm{H}_1 \Box \mathrm{H}_2}, \mathscr{P}^1_{\mathrm{H}_3 \Box \mathrm{H}_4}) \ge \min_{u=1,2,\ v=3,4} \{d_{B\min}(\mathscr{P}^0_{\mathrm{H_u}}, \mathscr{P}^0_{\mathrm{H_v}}), d_{\min}(\mathscr{P}^1_{\mathrm{H_u}}, \mathscr{P}^1_{\mathrm{H_v}})\}.$$

□

Definition 3.10 Let H_1 and H_2 two simple unweighted connected hyper graphs with Cartesian product $\mathrm{H}_1 \boxtimes \mathrm{H}_2$. Also we have $[\mathrm{H}_1 \boxtimes \mathrm{H}_2]_2$ be the 2-section of of $\mathrm{H}_1 \boxtimes \mathrm{H}_2$. Now for any $n \in \mathbb{Z}$, the clique complex of $[\mathrm{H}_1 \boxtimes \mathrm{H}_2]_2$ is defined as

$$\mathscr{C}\ \mathscr{L}_0([\mathrm{H}_1 \boxtimes \mathrm{H}_2]_2) \to \mathscr{C}\ \mathscr{L}_1([\mathrm{H}_1 \boxtimes \mathrm{H}_2]_2) \to \cdot \to \mathscr{C}\ \mathscr{L}_\mathrm{n}([\mathrm{H}_1 \boxtimes \mathrm{H}_2]_2) = [\mathrm{H}_1 \boxtimes \mathrm{H}_2]_2$$

where $\mathscr{C}\ \mathscr{L}_i([\mathrm{H}_1 \boxtimes \mathrm{H}_2]_2) = \sum_{\mathrm{j}=1}^{\mathrm{i}} \mathrm{k}'_\mathrm{j}$, k'_j is the jth skeleton of clique complex. Here the ith complex is the set of all simplices of dimension less than or equal to j. Here also filtration is from 0 to ∞. Which means adding vertices at the parameter, $\zeta = 0$, edges at $\zeta = 1$ and so on.

Definition 3.11 Let $\mathrm{H}_1, \mathrm{H}_2, \ldots, \mathrm{H_n}$ are simple unweighted connected hyper graphs with Cartesian product $\mathrm{H}_1 \boxtimes \mathrm{H}_2 \boxtimes \cdots \boxtimes \mathrm{H_n}$. Also we have $[\mathrm{H}_1 \boxtimes \mathrm{H}_2 \boxtimes \cdots \boxtimes \mathrm{H_n}]_2$ be the 2-section of of $\mathrm{H}_1 \boxtimes \mathrm{H}_2 \boxtimes \cdots \boxtimes \mathrm{H_n}$. Now for any $n \in \mathbb{Z}$, the clique complex of $[\mathrm{H}_1 \boxtimes \mathrm{H}_2 \boxtimes \cdots \boxtimes \mathrm{H_n}]_2$ is defined as

$$\mathfrak{CL}_0([\mathscr{H}_1 \boxtimes \mathrm{H}_2 \boxtimes \cdots \boxtimes \mathrm{H_n}]_2) \to \mathscr{C}\ \mathscr{L}_1([\mathrm{H}_1 \boxtimes \mathrm{H}_2 \boxtimes \cdots \boxtimes \mathrm{H_n}]_2) \to \cdots \to \mathscr{C}\ \mathscr{L}_\mathrm{n}([\mathrm{H}_1 \boxtimes \mathrm{H}_2 \boxtimes \cdots \boxtimes \mathrm{H_n}]_2)$$

with $\mathscr{C}\ \mathscr{L}_n([\mathrm{H}_1 \boxtimes \mathrm{H}_2 \boxtimes \cdots \boxtimes \mathrm{H_n}]_2) = [\mathrm{H}_1 \boxtimes \mathrm{H}_2 \boxtimes \cdots \boxtimes \mathrm{H_n}]_2$ where, $\mathscr{C}\ \mathscr{L}_i([\mathrm{H}_1 \boxtimes \mathrm{H}_2 \boxtimes \cdots \boxtimes \mathrm{H_n}]_2) = \sum_{\mathrm{j}=1}^{\mathrm{i}} \mathrm{K}'_{\mathrm{nj}}$, K'_{nj} is the jth skeleton of clique complex. Here the ith complex is the set of all simplices o dimension less than or equal to j. Our filtration starting from 0 to ∞. That is adding vertices at $\zeta' = 0$, edges at $\zeta' = 1$ and so on. Where ζ' is the parameter of filtration.

Theorem 3.12 *Let* H_1 *and* H_2 *two simple unweighted connected hyper graphs with strong product* $\mathrm{H}_1 \boxtimes \mathrm{H}_2$. *If* $[\mathrm{H}_1 \boxtimes \mathrm{H}_2]_2$ *be the* 2*-section of of* $\mathrm{H}_1 \boxtimes \mathrm{H}_2$, *then*

$$bcd_0([\mathrm{H}_1 \boxtimes \mathrm{H}_2]_2) = \{\mathrm{I} \cap \mathrm{J} \mid \mathrm{I} \in \mathrm{bcd}_0([\mathrm{H}_1]_2), \mathrm{J} \in \mathrm{bcd}_0([\mathrm{H}_2]_2)\}.$$

Proof When considering 0-dimensional persistent homology, the aim is to identify connected components at each stage of filtration. Consequently, in both the Cartesian product and the strong product of hyper graphs, the collection of zero-dimensional barcodes remains the same. Hence by theorem 3.5 we can conclude,

$$bcd_0([\mathrm{H}_1 \boxtimes \mathrm{H}_2]_2) = \{\mathrm{I} \cap \mathrm{J} \mid \mathrm{I} \in \mathrm{bcd}_0([\mathrm{H}_1]_2), \mathrm{J} \in \mathrm{bcd}_0([\mathrm{H}_2]_2)\}.$$

□

Theorem 3.13 *Let* $\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3$, and $\mathcal{H}_4$ *be the filtrations of hyper graphs* $\mathrm{H}_1, \mathrm{H}_2, \mathrm{H}_3$ and H_4 *respectively with* 0*-dimensional persistence diagrams* $\mathscr{P}^0_{H_1}, \mathscr{P}^0_{H_2}, \mathscr{P}^0_{H_3}$ and $\mathscr{P}^0_{H_4}$. *If* $\mathcal{H}_1 \boxtimes \mathcal{H}_2, \mathcal{H}_3 \boxtimes \mathcal{H}_4$ *be the filtrations of Cartesian products* $\mathrm{H}_1 \boxtimes \mathrm{H}_2$ and $\mathrm{H}_3 \boxtimes \mathrm{H}_4$ *with* 0*-dimensional persistence diagrams* $\mathscr{P}^0_{\mathrm{H}_1\boxtimes\mathrm{H}_2}, \mathscr{P}^0_{\mathrm{H}_3\boxtimes\mathrm{H}_4}$, *then*

$$d_B(\mathscr{P}^0_{\mathrm{H}_1\boxtimes\mathrm{H}_2}, \mathscr{P}^0_{\mathrm{H}_3\boxtimes\mathrm{H}_4}) \geq \min_{v=3,4}\{d_{B\min}(\mathscr{P}^0_{\mathrm{H}_1}, \mathscr{P}^0_{\mathrm{H}_v}), d_{B\min}(\mathscr{P}^0_{\mathrm{H}_2}, \mathscr{P}^0_{\mathrm{H}_v})\}.$$

Proof We have

$$bcd_0([\mathrm{H}_1 \boxtimes \mathrm{H}_2]_2) = \{\mathrm{I} \cap \mathrm{J} \mid \mathrm{I} \in \mathrm{bcd}_0([\mathrm{H}_1]_2), \mathrm{J} \in \mathrm{bcd}_0([\mathrm{H}_2]_2)\}.$$

Hence by theorem 3.7,

$$d_B(\mathscr{P}^0_{\mathrm{H}_1\boxtimes\mathrm{H}_2}, \mathscr{P}^0_{\mathrm{H}_3\boxtimes\mathrm{H}_4}) \geq \min_{v=3,4}\{d_{B\min}(\mathscr{P}^0_{\mathrm{H}_1}, \mathscr{P}^0_{\mathrm{H}_v}), d_{B\min}(\mathscr{P}^0_{\mathrm{H}_2}, \mathscr{P}^0_{\mathrm{H}_v})\}.$$

Theorem 3.14 *Let* $\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3$, and $\mathcal{H}_4$ *be the filtrations of hyper graphs* $\mathrm{H}_1, \mathrm{H}_2, \mathrm{H}_3$ and H_4 *respectively with* 1*-dimensional persistence diagrams* $\mathscr{P}^1_{\mathrm{H}_1}, \mathscr{P}^1_{\mathrm{H}_2}, \mathscr{P}^1_{\mathrm{H}_3}$ and $\mathscr{P}^1_{\mathrm{H}_4}$. *If* $\mathcal{H}_1 \boxtimes \mathcal{H}_2, \mathcal{H}_3 \boxtimes \mathcal{H}_4$ *be the filtrations of strong products* $\mathrm{H}_1 \boxtimes \mathrm{H}_2$ and $\mathrm{H}_3 \boxtimes \mathrm{H}_4$ *with* 1*-dimensional persistence diagrams* $\mathscr{P}^1_{\mathrm{H}_1\boxtimes\mathrm{H}_2}, \mathscr{P}^{'}_{\mathrm{H}_3\boxtimes\mathrm{H}_4}$ *then*

$$d_B(\mathscr{P}^1_{\mathrm{H}_1\boxtimes\mathrm{H}_2}, \mathscr{P}^1_{\mathrm{H}_3\boxtimes\mathrm{H}_4}) \geq \min_{v=3,4}\{d_{\min}(\mathscr{P}^1_{\mathrm{H}_1}, \mathscr{P}^1_{\mathrm{H}_v}), d_{\min}(\mathscr{P}^1_{\mathrm{H}_2}, \mathscr{P}^1_{\mathrm{H}_v})\}.$$

Proof Consider the two sections of hyper graphs $\mathrm{H}_1, \mathrm{H}_2, \mathrm{H}_3$ and H_4. Let $\{v_1, v_2, \ldots, v_n\}$ be the vertices of $[H_1]_2$ and $\{v'_1, v'_2, \ldots, v'_n\}$ be the vertices of $[\mathrm{H}_2]_2$. Then for any $v'_j \in [\mathrm{H}_2]_2$ the $[\mathrm{H}_1]_2$ fibre in $[\mathrm{H}_1 \boxtimes \mathrm{H}_2]_2$ is defined as

$$[\mathrm{H}_1]_2\mathrm{v}'_\mathrm{j} = \{(\mathrm{v_i}, \mathrm{v'_j}) | \mathrm{v_i} \in [\mathrm{H}_1]_2, \mathrm{i} = 1, 2, \ldots, \mathrm{n}\}, \mathrm{j} = 1, 2, \ldots, \mathrm{n}$$

and for any $v_j \in [\mathrm{H}_1]_2$ the $[\mathrm{H}_2]_2$ fibre in$n$ copies of $[\mathrm{H}_1]_2$ is defined as

$$v_j[\mathrm{H}_2]_2 = \{(\mathrm{v_j}, \mathrm{v'_i}) | \mathrm{v_i} \in [\mathrm{H}_1]_2, \mathrm{i} = 1, 2, \ldots, \mathrm{n}\}, \mathrm{j} = 1, 2, \ldots, \mathrm{n}.$$

Clearly $[\mathrm{H}_1]_2$ fibre is isomorphic to $[\mathrm{H}_1]_2$ and $[\mathrm{H}_2]_2$ fibre is isomorphic to $[\mathrm{H}_2]_2$. So there will be n copies of $[\mathrm{H}_1]_2$ and n copies of $[\mathrm{H}_2]_2$ in n copies of $[\mathrm{H}_1]_2$. Which means all the 1-dimensional holes in H_1 and H_2 will be there in $\mathrm{H}_1 \boxtimes \mathrm{H}_2$ n times. There won't be any other loops in strong product of hyper graphs. So

$$d_B(\mathscr{P}^1_{H_1 \boxtimes H_2}, \mathscr{P}^1_{H_3 \boxtimes H_4}) \geq \min_{\nu=3,4}\{d_{\min}(\mathscr{P}^1_{H_1}, \mathscr{P}^1_{H_\nu}), d_{\min}(\mathscr{P}^1_{H_2}, \mathscr{P}^1_{H_\nu})\}.$$

□

In the case of weighted hyper graph, we are doing filtration on hyper edge weights. Here we have to consider only 0 and 1 dimensional holes. While doing filtration, for any particular weight $\delta > 0$ the number of intervals $(0, \delta)$ in 0-dimensional barcode will represents the number of vertices in a hyper edge. If there are k number of $(0, \delta)$ intervals, then the corresponding hyper edge will contain $(k+1)$ vertices. If we are comparing two hyper graphs, we can compare 0-dimensional persistence diagram which will give the idea of connected components and if that bottle neck distance is greater than zero, we can say that these two hyper graphs not topologically similar. Additionally, we are using the same concept 2-section of the weighted hypergraph as in the unweighted case. Adding the simplex of 2-section which corresponds to the hyper edge according to their weight as parameter.

Definition 3.15 Let $\mathbf{H} = (\mathbf{V}_H, \mathbf{E}_H)$ be a simple connected weighted hyper graph with weight function $\mathbf{W}_H : \mathbf{E}_H \to \mathbb{R}$. Consider the 2-section $[\mathbf{H}]_2 = (\mathbf{V}_{[H]_2}, \mathbf{E}_{[H]_2})$ of $\mathbf{H}$. For any $\varepsilon > 0$ the 1-skeleton $([\mathbf{H}]_2)_\varepsilon = ((\mathbf{V}_{[H]_2})_\varepsilon, (\mathbf{E}_{[H]_2})_\varepsilon)$ is defined as the sub graph of $[\mathbf{H}]_2$ where $(\mathbf{V}_{[H]_2})_\varepsilon = \mathbf{V}_H$ and its edge set $(\mathbf{E}_{[H]_2})_\varepsilon) \in \mathbf{E}_{[H]_2}$ includes only the two sections of hyper edges whose weight is less than or equal to ε. Then for any $\varepsilon \in \mathbb{R}$, we define the Vietoris-Rips complex $\mathbf{H}_\nu$ as the clique complex of 1-skeleton of $([\mathbf{H}]_2)_\varepsilon$,, $\mathcal{CL}([\mathbf{H}]_2)_\varepsilon$, and the vietoris-Rips filtration in weighted hypergraph is defined as

$$\{\mathcal{CL}([\mathbf{H}]_2)_\varepsilon \to \mathcal{CL}([\mathbf{H}]_2)_{\varepsilon'}\}_{0 \leq \varepsilon \leq \varepsilon'}.$$

The filtration starts with vertex set and the hyper edge weight is assumed to be 0 to ∞. For each step two sections of hyper edges are added and the corresponding complex is found.

Definition 3.16 Let $\mathbf{H}_1 = (\mathbf{V}_{H_1}, \mathbf{E}_{H_1})$ and $\mathbf{H}_2 = (\mathbf{V}_{H_2}, \mathbf{E}_{H_2})$ be two edge weighted simple connected hyper graphs with weight functions $\mathbf{W}_{\mathbf{H}_1}$ and $\mathbf{W}_{\mathbf{H}_2}$ respectively. Then the Cartesian product $\mathbf{H}_1 \square \mathbf{H}_2$ of $\mathbf{H}_1$ and $\mathbf{H}_2$ is defined as $\mathbf{H}_1 \square \mathbf{H}_2 = (\mathbf{V}_{\mathbf{H}_1 \square \mathbf{H}_2}, \mathbf{E}_{\mathbf{H}_1 \square \mathbf{H}_2})$ where

$$\mathbf{V}_{\mathbf{H}_1 \square \mathbf{H}_2} = \mathbf{V}_{\mathbf{H}_1} \times \mathbf{V}_{\mathbf{H}_2}$$

and

$$\mathbf{E}_{\mathbf{H}_1 \square \mathbf{H}_2} = \{\{x\} \times e : x \in \mathbf{V}_{H_1}, e \in \mathbf{E}_{H_2}\} \bigcup \{e \times \{y\} : e \in \mathbf{E}_{H_1}, y \in \mathbf{V}_{H_2}\}$$

with weight function $\mathbf{W}_{\mathbf{H}_1 \square \mathbf{H}_2} : \mathbf{E}_{\mathbf{H}_1 \square \mathbf{H}_2} \to \mathbb{R}$ defined by,

$$\mathbf{W}_{\mathbf{H}_1 \square \mathbf{H}_2}(x, e_1) = \mathbf{W}_{\mathbf{H}_2}(e_1) \text{ for all } e_1 \in \mathbf{E}_{H_2}$$

and

$$\mathbf{W}_{\mathbf{H}_1 \square \mathbf{H}_2}(e_2, y) = \mathbf{W}_{\mathbf{H}_1}(e_2) \text{ for all } e_2 \in \mathbf{E}_{H_1}.$$

Definition 3.17 Let $\mathbf{H_1} = (\mathbf{V}_{H_1}, \mathbf{E}_{H_1})$ and $\mathbf{H_2} = (\mathbf{V}_{H_2}, \mathbf{E}_{H_2})$ be weighted simple, connected and weighted graphs with weight functions $\mathbf{W}_{\mathbf{H}_1}$ and $\mathbf{W}_{\mathbf{H}_2}$ respectively. Consider the Cartesian product $\mathbf{H}_1 \Box \mathbf{H}_2$ of weighted hyper graphs $\mathbf{H_1}$ and $\mathbf{H_2}$ with its 2-section $[\mathbf{H}_1 \Box \mathbf{H}_2]_2$. For any $\upsilon > 0$ the 1-skeleton $(\mathbf{H}_1 \Box \mathbf{H}_2)_\upsilon = ((\mathbf{V}_{\mathbf{H}_1 \Box \mathbf{H}_2})_\upsilon, (\mathbf{E}_{\mathbf{H}_1 \Box \mathbf{H}_2})_\upsilon)$ is defined as the sub graph of $\mathbf{H}_1 \Box \mathbf{H}_2$ where $(\mathbf{V}_{\mathbf{H}_1 \Box \mathbf{H}_2})_\upsilon = \mathbf{V}_{H_1 \Box \mathbf{H}_2}$ and its edge set $(\mathbf{E}_{[\mathbf{H}_1 \Box \mathbf{H}_2]_2})_\upsilon \in \mathbf{E}_{[\mathbf{H}_1 \Box \mathbf{H}_2]_2}$ includes only the two sections of hyper edges whose weight is less than or equal to υ. Then for any $\upsilon \in \mathbb{R}$, we define the Vietoris-Rips complex $\mathbf{H}_1 \Box \mathbf{H}_{2\nu}$ as the clique complex of 1-skeleton of $([\mathbf{H}_1 \Box \mathbf{H}_2]_2)_\upsilon, (\mathcal{CL}([\mathbf{H}_1 \Box \mathbf{H}_2]_2)_\upsilon$, and the vietoris-Rips filtration is defined as

$$\{\mathcal{CL}([\mathbf{H}_1 \Box \mathbf{H}_2]_2)_\upsilon \rightarrow \mathcal{CL}([\mathbf{H}_1 \Box \mathbf{H}_2]_2)_{\upsilon'}\}_{0 \leq \upsilon \leq \upsilon'}.$$

The filtration starts with vertex set and the hyper edge weight is assumed to be 0 to ∞. For each step two sections of hyper edges are added and the corresponding complex is found.

Theorem 3.18 *Let $\mathcal{H}'_1$ and $\mathcal{H}'_2$ are the filtrations of two simple weighted connected hyper graphs $\mathbf{H}_1$ and $\mathbf{H}_2$ with Cartesian product $\mathbf{H}_1 \Box \mathbf{H}_2$. If $\mathcal{H}'_1 \Box \mathcal{H}'_2$ be the filtration of $\mathbf{H}_1 \Box \mathbf{H}_2$, then*

$$bcd_0(\mathcal{H}'_1 \Box \mathcal{H}'_2) = \{\mathbf{I} \cap \mathbf{J} \mid \mathbf{I} \in bcd_0(\mathcal{H}'_1), \mathbf{J} \in bcd_0(\mathcal{H}'_2)\}.$$

Proof $\mathcal{H}'_1$ and $\mathcal{H}'_2$ are the filtrations of two simple weighted connected hyper graphs $\mathbf{H}_1$ and $\mathbf{H}_2$ with Cartesian product $\mathbf{H}_1 \Box \mathbf{H}_2$ and $[\mathbf{H}_1]_2, [\mathbf{H}_2]_2$ are 2 sections of $\mathbf{H}_1$ and $\mathbf{H}_2$ respectively. Let

$$bcd_0(\mathcal{H}'_1) = \{(0, \mathfrak{a}_1), (0, \mathfrak{a}_2) \dots (0, \mathfrak{a}_n) = (0, \infty)\} \text{ with } \mathfrak{a}_1 \leq \mathfrak{a}_2 \leq \dots \leq \mathfrak{a}_n$$

and

$$bcd_0(\mathcal{H}'_2) = \{(0, \mathfrak{b}_1), (0, \mathfrak{b}_2) \dots, (0, \mathfrak{b}_n) = (0, \infty)\} \text{ with } \mathfrak{b}_1 \leq \mathfrak{b}_2 \leq \dots \leq \mathfrak{b}_n.$$

Here each $\mathfrak{a}_i$ and $\mathfrak{b}_j$ represents weight of each hyper edge and intervals $(0, \mathfrak{a}_i)$ and $(0, \mathfrak{b}_j)$ may repeat according to the number of vertices in each edge for all $i, j = 1, 2, \dots, n$.

Without loss of generality, assume $\mathfrak{a}_2 \leq \mathfrak{a}_k, \mathfrak{b}_j$ for all $k = 3, 4, \dots, n, j = 1, 2, \dots, n$, there will be again n number of $(0, \mathfrak{a}_2)$ intervals in $bcd_0(\mathcal{H}'_1 \Box \mathcal{H}'_2)$. It can be written as

$$(0, \mathfrak{a}_2) = (0, \mathfrak{a}_2) \cap (0, \mathfrak{b}_j) \text{ for all } j = 1, 2, \dots, n.$$

If $\mathfrak{b}_1 \leq \mathfrak{a}_m, \mathfrak{b}_p$ for all $m = 3, 4, \dots, n$ and $p = 2, 3, \dots, n$, then there we have to consider only $(n-2)$ components of $\mathbf{H}_1$ and so that $(n-2)$ times the interval $(0, \mathfrak{b}_1)$ will be there in $bcd_0(\mathcal{H}'_1 \Box \mathcal{H}'_2)$. It can be represented as

$$(0, \mathfrak{b}_1) = (0, \mathfrak{a}_m) \cap (0, \mathfrak{b}_1) \text{ for all } m = 3, 4, \dots, n.$$

Continuing like this we will get

$$bcd_0(\mathcal{H}_1'\square\mathcal{H}_2') = \{\mathbf{I}\cap\mathbf{J} \mid \mathbf{I}\in bcd_0(\mathcal{H}_1'), \mathbf{J}\in bcd_0(\mathcal{H}_2')\}.$$

□

Here also in Vietoris-Rips filtration, we are considering parameters from 0 to ∞. Hence the minimum value of each interval in zero dimensional barcode will be zero. In Bottleneck distance we are taking maximum distance. So in the case of zero dimensional persistence diagrams in weighted hyper graph also we have to consider the fallowing terms,

$$d_{B\min}(\mathbf{P}^k, \mathbf{Q}^k) = \min_{i,j=1,2,\ldots,k}\{|\mathfrak{b}_{i_l} - \mathfrak{b}'_{j_t}|,\ l,t = 1,2,\ldots,n\}$$
$$d_{B\min}(\mathbf{P}^k, \mathbf{Q}'^k) = \min_{i,j=1,2,\ldots,k}\{|\mathfrak{b}_{i_l} - \mathfrak{d}'_{j_t}|,\ l,t = 1,2,\ldots,n\}$$
$$d_{B\min}(\mathbf{P}'^k, \mathbf{Q}^k) = \min_{i,j=1,2,\ldots,k}\{|\mathfrak{b}'_{j_t} - \mathfrak{d}_{i_q}|,\ q,t = 1,2,\ldots,n\}$$

and

$$d_{B\min}(\mathbf{P}'^k, \mathbf{Q}'^k) = \min_{i,j=1,2,\ldots,k}\{|\mathfrak{b}'_{j_t} - \mathfrak{d}'_{i_q}|,\ q,t = 1,2,\ldots,n\},\ \text{for } k = 0,1.$$

Theorem 3.19 *Let* $\mathcal{H}_1', \mathcal{H}_2', \mathcal{H}_3'$, and $\mathcal{H}_4'$ *are the filtrations of weighted hyper graphs* $\mathbf{H}_1, \mathbf{H}_2, \mathbf{H}_3$ *and* $\mathbf{H}_4$ *respectively with* 0*-dimensional persistence diagrams* $\mathcal{P}^0_{\mathbf{H}_1}, \mathcal{P}^0_{\mathbf{H}_2}, \mathcal{P}^0_{\mathbf{H}_3}$ and $\mathcal{P}^0_{\mathbf{H}_4}$. *If* $\mathbf{P}^0_{\mathbf{H}_1\square\mathbf{H}_2}$ *and* $\mathbf{P}^0_{\mathbf{H}_3\square\mathbf{H}_4}$ *are the* 0*-dimensional persistence diagrams of Cartesian products* $\mathbf{H}_1\square\mathbf{H}_2$ *and* $\mathbf{H}_3\square\mathbf{H}_4$ *then*

$$d_B(\mathbf{P}^0_{\mathbf{H}_1\square\mathbf{H}_2}, \mathbf{P}^0_{\mathbf{H}_3\square\mathbf{H}_4}) \geq \min_{\nu=3,4}\{d_{B\min}(\mathbf{P}^0_{\mathbf{H}_1}, \mathbf{P}^0_{\mathbf{H}_\nu}), d_{B\min}(\mathbf{P}^0_{\mathbf{H}_2}, \mathbf{P}^0_{\mathbf{H}_\nu})\}.$$

Proof We have

$$bcd_0(\mathcal{H}_1'\square\mathcal{H}_2) = \{\mathbf{I}\cap\mathbf{J} \mid \mathbf{I}\in bcd_0(\mathcal{H}_1'), \mathbf{J}\in bcd_0(\mathcal{H}_2')\}$$

and

$$bcd_0(\mathcal{H}_3'\square\mathcal{H}_4') = \{\mathbf{I}'\cap\mathbf{J}' \mid \mathbf{I}'\in bcd_0(\mathcal{H}_3'), J'\in bcd_0(\mathcal{H}_4')\}.$$

Which means the 0-dimensional of Cartesian product of two weighted hyper graphs is same as the Cartesian product of topological spaces. So by theorem 3.6,

$$d_B(\mathbf{P}^0_{\mathbf{H}_1\square\mathbf{H}_2}, \mathbf{P}^0_{\mathbf{H}_3\square\mathbf{H}_4}) \geq \min_{\nu=3,4}\{d_{B\min}(\mathbf{P}^0_{\mathbf{H}_1}, \mathbf{P}^0_{\mathbf{H}_\nu}), d_{B\min}(\mathbf{P}^0_{\mathbf{H}_2}, \mathbf{P}^0_{\mathbf{H}_\nu})\}.$$

□

Here also in weighted hyper graphs, the minimum value intervals in one dimensional persistence diagrams will be always greater than zero since at zero there won't be any one dimensional hole. Also when considering the 2-section, we are giving weights for each k-cliques.

Theorem 3.20 *Let* $\mathcal{H}'_1, \mathcal{H}'_2, \mathcal{H}'_3$, and $\mathcal{H}'_4$ *are the filtrations of weighted hyper graphs* $\mathbf{H}_1, \mathbf{H}_2, \mathbf{H}_3$ *and* $\mathbf{H}_4$ *respectively with* 1*-dimensional persistence diagrams* $\mathcal{P}^1_{\mathbf{H}_1}, \mathcal{P}^1_{\mathbf{H}_2}, \mathcal{P}^1_{\mathbf{H}_3}$ and $\mathcal{P}^1_{\mathbf{H}_4}$. *If* $\mathcal{H}'_1 \Box \mathcal{H}'_2, \mathcal{H}'_3 \Box \mathcal{H}'_4$ *be the filtrations of Cartesian products* $\mathbf{H}_1 \Box \mathbf{H}_2$ and $\mathbf{H}_3 \Box \mathbf{H}_4$ *with* 1*-dimensional persistence diagrams* $\mathcal{P}^1_{\mathbf{H}_1 \Box \mathbf{H}_2}, \mathcal{P}^1_{\mathbf{H}_3 \Box \mathbf{H}_4}$, *then*

$$d_B(\mathcal{P}^1_{\mathbf{H}_1 \Box \mathbf{H}_2}, \mathcal{P}^1_{\mathbf{H}_3 \Box \mathbf{H}_4}) \geq \min_{u=1,2,\ v=3,4} \{d_{B\min}(\mathcal{P}^0_{\mathbf{H}_u}, \mathcal{P}^0_{\mathbf{H}_v}), d_{\min}(\mathcal{P}^1_{\mathbf{H}_u}, \mathcal{P}^1_{\mathbf{H}_v})\}.$$

Proof Consider the two sections of weighted hyper graphs $\mathbf{H}_1, \mathbf{H}_2, \mathbf{H}_3$ and $\mathbf{H}_4$. Let $\{\mathfrak{v}_1, \mathfrak{v}_2, \ldots, \mathfrak{v}_n\}$ be the vertices of $[H_1]_2$ and $\{\mathfrak{v}'_1, \mathfrak{v}'_2, \ldots, \mathfrak{v}'_n\}$ be the vertices of $[\mathbf{H}_2]_2$. Then for any $\mathfrak{v}'_j \in [\mathbf{H}_2]_2$ the $[\mathbf{H}_1]_2$ fibre in $[\mathbf{H}_1 \Box \mathbf{H}_2]_2$ is defined as

$$[\mathbf{H}_1]_2 \mathfrak{v}'_J = \{(\mathfrak{v}_i, \mathfrak{v}'_j) | \mathfrak{v}_i \in [\mathbf{H}_1]_2, i = 1, 2, \ldots, n\}, j = 1, 2, \ldots, n$$

and for any $\mathfrak{v}_j \in [\mathbf{H}_1]_2$ the $[\mathbf{H}_2]_2$ fibre in n copies of $[\mathbf{H}_1]_2$ is defined as

$$\mathfrak{v}_J [\mathbf{H}_2]_2 = \{(\mathfrak{v}_j, \mathfrak{v}'_i) | \mathfrak{v}_i \in [\mathbf{H}_1]_2, i = 1, 2, \ldots, n\}, j = 1, 2, \ldots, n.$$

Clearly $[\mathbf{H}_1]_2$ fibre is isomorphic to $[\mathbf{H}_1]_2$ and $[\mathbf{H}_2]_2$ fibre is isomorphic to $[\mathbf{H}_2]_2$. So there will be n copies of $[\mathbf{H}_1]_2$ and n copies of $[\mathbf{H}_2]_2$ in n copies of $[\mathbf{H}_1]_2$. Which means all the 1-dimensional holes in $\mathbf{H}_1$ and $\mathbf{H}_2$ will be there in $\mathbf{H}_1 \Box \mathbf{H}_2$ n times. Some other loops also will be there and which will be of the form $(t, \infty), 0 \leq t \leq \infty$ with $t = \max\{t_1, t_2\}$ where $(0, t_1) \in bcd_0(\mathcal{H}'_1)$ and $(0, t_2) \in bcd_0(\mathcal{H}'_2)$. If we are considering all these cases in $\mathbf{H}_3$ and $\mathbf{H}_4$, we can conclude that

$$d_B(\mathcal{P}^1_{\mathbf{H}_1 \Box \mathbf{H}_2}, \mathcal{P}^1_{\mathbf{H}_3 \Box \mathbf{H}_4}) \geq \min_{u=1,2,\ v=3,4} \{d_{B\min}(\mathcal{P}^0_{\mathbf{H}_u}, \mathcal{P}^0_{\mathbf{H}_v}), d_{\min}(\mathcal{P}^1_{\mathbf{H}_u}, \mathcal{P}^1_{\mathbf{H}_v})\}.$$

□

Definition 3.21 Let $\mathbf{H}_1 = (\mathbf{V}_{H_1}, \mathbf{E}_{H_1})$ and $\mathbf{H}_2 = (\mathbf{V}_{H_2}, \mathbf{E}_{H_2})$ be two edge weighted simple connected hyper graphs with weight functions $\mathbf{W}_{\mathbf{H}_1}$ and $\mathbf{W}_{\mathbf{H}_2}$ respectively. Then the strong product $\mathbf{H}_1 \boxtimes \mathbf{H}_2$ of $\mathbf{H}_1$ and $\mathbf{H}_2$ is defined as $\mathbf{H}_1 \boxtimes \mathbf{H}_2 = (\mathbf{V}_{\mathbf{H}_1 \boxtimes \mathbf{H}_2}, \mathbf{E}_{\mathbf{H}_1 \boxtimes \mathbf{H}_2})$ with weight function $\mathbf{W}_{\mathbf{H}_1 \boxtimes \mathbf{H}_2}$ where

$$\mathbf{V}_{\mathbf{H}_1 \boxtimes \mathbf{H}_2} = \mathbf{V}_{\mathbf{H}_1} \times \mathbf{V}_{\mathbf{H}_2}$$

and a subset $\mathfrak{e} = \{(\mathfrak{v}_1, \mathfrak{v}'_1), (\mathfrak{v}_2, \mathfrak{v}'_2), \ldots, (\mathfrak{v}_n, \mathfrak{v}'_n)\}$ of $\mathbf{V}_{\mathbf{H}_1} \times \mathbf{V}_{\mathbf{H}_2}$ is an edge in edge set $\mathbf{E}_{\mathbf{H}_1 \boxtimes \mathbf{H}_2}$ of $\mathbf{H}_1 \boxtimes \mathbf{H}_2$ if,

1. $\{\mathfrak{v}_1, \mathfrak{v}_2, \ldots, \mathfrak{v}_n\}$ is an edge of $\mathbf{H}_1$ and $\mathfrak{v}'_1 = \mathfrak{v}'_2 = \cdots = \mathfrak{v}'_n \in \mathbf{V}_{\mathbf{H}_2}$, or
2. $\{\mathfrak{v}'_1, \mathfrak{v}'_2, \ldots, \mathfrak{v}'_n\}$ is an edge of $\mathbf{H}_2$ and $\mathfrak{v}_1 = \mathfrak{v}_2 = \cdots = \mathfrak{v}_n \in \mathbf{V}_{\mathbf{H}_1}$, or
3. $\{\mathfrak{v}_1, \mathfrak{v}_2, \ldots, \mathfrak{v}_n\}$ is an edge of $\mathbf{H}_1$ and there is an edge $\mathfrak{f} \in \mathbf{E}_{H_2}$ such that $\{\mathfrak{v}'_1, \mathfrak{v}'_2, \ldots, \mathfrak{v}'_n\}$ is a multi set of elements of $\mathfrak{f}$, and $\mathfrak{f} \subseteq \{\mathfrak{v}_1, \mathfrak{v}_2, \ldots, \mathfrak{v}_n\}$, or

4. $\{\mathfrak{v}'_1, \mathfrak{v}'_2, \ldots, \mathfrak{v}'_n\}$ is an edge of $\mathbf{H}_2$ and there is an edge $\mathfrak{f} \in \mathbf{E}_{H_1}$ such that $\{\mathfrak{v}_1, \mathfrak{v}_2, \ldots, \mathfrak{v}_n\}$ is a multi set of elements of $\mathfrak{f}$, and $\mathfrak{f} \subseteq \{\mathfrak{v}_1, \mathfrak{v}_2, \ldots, \mathfrak{v}_n\}$

with weight function $\mathbf{W}_{\mathbf{H}_1 \boxtimes \mathbf{H}_2} : \mathbf{E}_{\mathbf{H}_1 \boxtimes \mathbf{H}_2} \to \mathbb{R}$ defined by,

$$\mathbf{W}_{\mathbf{H}_1 \boxtimes \mathbf{H}_2}((\mathfrak{v}_1, \mathfrak{v}'_1), (\mathfrak{v}_2, \mathfrak{v}'_1), \ldots, (\mathfrak{v}_n, \mathfrak{v}'_1)) = \mathbf{W}_{\mathbf{H}_2}(\mathfrak{v}_1, \mathfrak{v}_2, \ldots, \mathfrak{v}_n) \text{ for all } (\mathfrak{v}_1, \mathfrak{v}_2, \ldots, \mathfrak{v}_n) \in \mathbf{E}_{H_1},$$
$$\mathbf{W}_{\mathbf{H}_1 \boxtimes \mathbf{H}_2}((\mathfrak{v}_1, \mathfrak{v}'_1), (\mathfrak{v}_1, \mathfrak{v}'_2), \ldots, (\mathfrak{v}_1, \mathfrak{v}'_n)) = \mathbf{W}_{\mathbf{H}_2}(\mathfrak{v}'_1, \mathfrak{v}'_2, \ldots, \mathfrak{v}'_n) \text{ for all } (\mathfrak{v}'_1, \mathfrak{v}'_2, \ldots, \mathfrak{v}'_n) \in \mathbf{E}_{H_2},$$
$$\mathbf{W}_{\mathbf{H}_1 \boxtimes \mathbf{H}_2}((\mathfrak{v}_1, \mathfrak{v}'_1), (\mathfrak{v}_1, \mathfrak{v}'_2), \ldots, (\mathfrak{v}_1, \mathfrak{v}'_n)) = \min\{\mathbf{W}_{\mathbf{H}_2}(\mathfrak{v}_1, \mathfrak{v}_2, \ldots, \mathfrak{v}_n), \mathbf{W}_{\mathbf{H}_1}(\mathfrak{f})\} \text{ if } \{\mathfrak{v}_1, \mathfrak{v}_2, \ldots, \mathfrak{v}_n\}$$

is an edge of $\mathbf{H}_1$ and $\mathfrak{f} \subseteq \{\mathfrak{v}_1, \mathfrak{v}_2, \ldots, \mathfrak{v}_n\}$ and

$$\mathbf{W}_{\mathbf{H}_1 \boxtimes \mathbf{H}_2}((\mathfrak{v}_1, \mathfrak{v}'_1), (\mathfrak{v}_1, \mathfrak{v}'_2), \ldots, (\mathfrak{v}_1, \mathfrak{v}'_n)) = \min\{\mathbf{W}_{\mathbf{H}_1}(\mathfrak{f}), \mathbf{W}_{\mathbf{H}_2}(\mathfrak{v}'_1, \mathfrak{v}'_2, \ldots, \mathfrak{v}'_n)\} \text{ if } \{\mathfrak{v}_1, \mathfrak{v}_2, \ldots, \mathfrak{v}_n\}$$

is an edge of $\mathbf{H}_2$ and $\mathfrak{f} \subseteq \{\mathfrak{v}'_1, \mathfrak{v}'_2, \ldots, \mathfrak{v}'_n\}$.

Definition 3.22 Let $\mathbf{H_1} = (\mathbf{V}_{H_1}, \mathbf{E}_{H_1})$ and $\mathbf{H_2} = (\mathbf{V}_{H_2}, \mathbf{E}_{H_2})$ be weighted simple, connected and weighted graphs with weight functions $\mathbf{W}_{\mathbf{H}_1}$ and $\mathbf{W}_{\mathbf{H}_2}$ respectively. Consider the strong product $\mathbf{H}_1 \boxtimes \mathbf{H}_2$ of weighted hyper graphs $\mathbf{H_1}$ and $\mathbf{H_2}$ with its 2-section $[\mathbf{H}_1 \boxtimes \mathbf{H}_2]_2$. For any $\upsilon > 0$ the 1-skeleton $(\mathbf{H}_1 \boxtimes \mathbf{H}_2)_\upsilon = ((\mathbf{V}_{\mathbf{H}_1 \boxtimes \mathbf{H}_2})_\upsilon, (\mathbf{E}_{\mathbf{H}_1 \boxtimes \mathbf{H}_2})_\upsilon)$ is defined as the sub graph of $\mathbf{H}_1 \boxtimes \mathbf{H}_2$ where $(\mathbf{V}_{\mathbf{H}_1 \boxtimes \mathbf{H}_2})_\upsilon = \mathbf{V}_{H_1 \boxtimes \mathbf{H}_2}$ and its edge set $(\mathbf{E}_{[\mathbf{H}_1 \boxtimes \mathbf{H}_2]_2})_\upsilon \in \mathbf{E}_{[\mathbf{H}_1 \boxtimes \mathbf{H}_2]_2}$ includes only the two sections of hyper edges whose weight is less than or equal to υ. Then for any $\upsilon \in \mathbb{R}$, we define the Vietoris-Rips complex $(\mathbf{H}_1 \boxtimes \mathbf{H}_2)_\nu$ as the clique complex of 1-skeleton of $([\mathbf{H}_1 \boxtimes \mathbf{H}_2]_2)_\upsilon, (\mathcal{CL}([\mathbf{H}_1 \boxtimes \mathbf{H}_2]_2)_\upsilon$, and the vietoris-Rips filtration is defined as

$$\{\mathcal{CL}([\mathbf{H}_1 \boxtimes \mathbf{H}_2]_2)_\upsilon \to \mathcal{CL}([\mathbf{H}_1 \boxtimes \mathbf{H}_2]_2)_{\upsilon'}\}_{0 \leq \upsilon \leq \upsilon'}.$$

The filtration starts with vertex set and the hyper edge weight is assumed to be 0 to ∞. For each step two sections of hyper edges are added and the corresponding complex is found.

Theorem 3.23 *Let* $\mathbf{H}_1$ *and* $\mathbf{H}_2$ *two simple weighted connected hyper graphs with strong product* $\mathbf{H}_1 \boxtimes \mathbf{H}_2$. *If* $[\mathbf{H}_1 \boxtimes \mathbf{H}_2]_2$ *be the* 2*-section of of* $\mathbf{H}_1 \boxtimes \mathbf{H}_2$, *then*

$$bcd_0([\mathbf{H}_1 \boxtimes \mathbf{H}_2]_2) = \{\mathbf{I} \cap \mathbf{J} \mid \mathbf{I} \in bcd_0([\mathbf{H}_1]_2), \mathbf{J} \in bcd_0([\mathbf{H}_2]_2)\}.$$

Proof When considering 0-dimensional persistent homology in weighted hyper graphs, in both the Cartesian product and the strong product of hyper graphs, the collection of zero-dimensional barcodes remains the same. Hence by theorem 3.15 we can conclude,

$$bcd_0([\mathbf{H}_1 \boxtimes \mathbf{H}_2]_2) = \{\mathbf{I} \cap \mathbf{J} \mid \mathbf{I} \in bcd_0([\mathbf{H}_1]_2), \mathbf{J} \in bcd_0([\mathbf{H}_2]_2)\}.$$

Theorem 3.24 *Let* $\mathcal{H}'_1, \mathcal{H}'_2, \mathcal{H}'_3$, and $\mathcal{H}'_4$ *are the filtrations of weighted hyper graphs* $\mathbf{H}_1, \mathbf{H}_2, \mathbf{H}_3$ *and* $\mathbf{H}_4$ *respectively with* 0*-dimensional persistence diagrams* $\mathcal{P}^0_{\mathbf{H}_1}, \mathcal{P}^0_{\mathbf{H}_2}, \mathcal{P}^0_{\mathbf{H}_3}$ and $\mathcal{P}^0_{\mathbf{H}_4}$. *If* $\mathbf{P}^0_{\mathbf{H}_1 \boxtimes \mathbf{H}_2}$ *and* $\mathbf{P}^0_{\mathbf{H}_3 \boxtimes \mathbf{H}_4}$ *are the* 0*-dimensional persistence diagrams of strong products* $\mathbf{H}_1 \boxtimes \mathbf{H}_2$ *and* $\mathbf{H}_3 \boxtimes \mathbf{H}_4$ *then*

$$d_B(\mathbf{P}^0_{\mathbf{H}_1 \boxtimes \mathbf{H}_2}, \mathbf{P}^0_{\mathbf{H}_3 \boxtimes \mathbf{H}_4}) \geq \min_{\nu=3,4}\{d_{B\min}(\mathbf{P}^0_{\mathbf{H}_1}, \mathbf{P}^0_{\mathbf{H}_\nu}), d_{B\min}(\mathbf{P}^0_{\mathbf{H}_2}, \mathbf{P}^0_{\mathbf{H}_\nu})\}.$$

Proof We have

$$bcd_0(\mathcal{H}_1' \boxtimes \mathcal{H}_2) = \{\mathbf{I} \cap \mathbf{J} \mid \mathbf{I} \in bcd_0(\mathcal{H}_1'), \mathbf{J} \in bcd_0(\mathcal{H}_2')\}$$

and

$$bcd_0(\mathcal{H}_3' \boxtimes \mathcal{H}_4') = \{\mathbf{I}' \cap \mathbf{J}' \mid \mathbf{I}' \in bcd_0(\mathcal{H}_3'), J' \in bcd_0(\mathcal{H}_4')\}.$$

Which means the 0-dimensional of strong product of two weighted hyper graphs is same as the Cartesian product of topological spaces. So by theorem 3.6,

$$d_B(\mathbf{P}^0_{\mathbf{H}_1 \boxtimes \mathbf{H}_2}, \mathbf{P}^0_{\mathbf{H}_3 \boxtimes \mathbf{H}_4}) \geq \min_{\nu=3,4}\{d_{B\min}(\mathbf{P}^0_{\mathbf{H}_1}, \mathbf{P}^0_{\mathbf{H}_\nu}), d_{B\min}(\mathbf{P}^0_{\mathbf{H}_2}, \mathbf{P}^0_{\mathbf{H}_\nu})\}.$$

Theorem 3.25 *Let* $\mathcal{H}_1', \mathcal{H}_2', \mathcal{H}_3'$, and $\mathcal{H}_4'$ *be the filtrations of weighted simple and connected hyper graphs* $\mathbf{H}_1, \mathbf{H}_2, \mathbf{H}_3$ and $\mathbf{H}_4$ *respectively with* 1*-dimensional persistence diagrams* $\mathcal{P}^1_{\mathbf{H}_1}, \mathcal{P}^1_{\mathbf{H}_2}, \mathcal{P}^1_{\mathbf{H}_3}$ and $\mathcal{P}^1_{\mathbf{H}_4}$. *If* $\mathcal{H}_1' \boxtimes \mathcal{H}_2', \mathcal{H}_3' \boxtimes \mathcal{H}_4'$ *be the filtrations of strong products* $\mathbf{H}_1 \boxtimes \mathbf{H}_2$ and $\mathbf{H}_3 \boxtimes \mathbf{H}_4$ *with* 1*-dimensional persistence diagrams* $\mathcal{P}^1_{\mathbf{H}_1 \boxtimes \mathbf{H}_2}, \mathcal{P}_{\mathbf{H}_3 \boxtimes \mathbf{H}_4}$, *then*

$$d_B(\mathcal{P}^1_{\mathbf{H}_1 \boxtimes \mathbf{H}_2}, \mathcal{P}^1_{\mathbf{H}_3 \boxtimes \mathbf{H}_4}) \geq \min_{\nu=3,4}\{d_{\min}(\mathcal{P}^1_{\mathbf{H}_1}, \mathcal{P}^1_{\mathbf{H}_\nu}), d_{\min}(\mathcal{P}^1_{\mathbf{H}_2}, \mathcal{P}^1_{\mathbf{H}_\nu})\}.$$

Proof Consider the two sections of weighted hyper graphs $\mathbf{H}_1, \mathbf{H}_2, \mathbf{H}_3$ and $\mathbf{H}_4$. Let $\{\mathfrak{v}_1, \mathfrak{v}_2, \ldots, \mathfrak{v}_n\}$ be the vertices of $[\mathbf{H}_1]_2$ and $\{\mathfrak{v}_1', \mathfrak{v}_2', \ldots, \mathfrak{v}_n'\}$ be the vertices of $[\mathbf{H}_2]_2$. Then for any $\mathfrak{v}_j' \in [\mathbf{H}_2]_2$ the $[\mathbf{H}_1]_2$ fibre in $[\mathbf{H}_1 \boxtimes \mathbf{H}_2]_2$ is defined as

$$[\mathbf{H}_1]_2\mathfrak{v}_J' = \{(\mathfrak{v}_i, \mathfrak{v}_j')|\mathfrak{v}_i \in [\mathbf{H}_1]_2, i = 1, 2, \ldots, n\}, j = 1, 2, \ldots, n$$

and for any $\mathfrak{v}_j \in [\mathbf{H}_1]_2$ the $[\mathbf{H}_2]_2$ fibre in$n$ copies of $[\mathbf{H}_1]_2$ is defined as

$$\mathfrak{v}_J[\mathbf{H}_2]_2 = \{(\mathfrak{v}_j, \mathfrak{v}_i')|\mathfrak{v}_i \in [\mathbf{H}_1]_2, i = 1, 2, \ldots, n\}, j = 1, 2, \ldots, n.$$

Clearly $[\mathbf{H}_1]_2$ fibre is isomorphic to $[\mathbf{H}_1]_2$ and $[\mathbf{H}_2]_2$ fibre is isomorphic to $[\mathbf{H}_2]_2$. So there will be n copies of $[\mathbf{H}_1]_2$ and n copies of $[\mathbf{H}_2]_2$ in n copies of $[\mathbf{H}_1]_2$. Which means all the 1-dimensional holes in $\mathbf{H}_1$ and $\mathbf{H}_2$ will be there in $\mathbf{H}_1 \boxtimes \mathbf{H}_2$ n times. There won't be any other loops in strong product of hyper graphs. So

$$d_B(\mathcal{P}^1_{\mathbf{H}_1 \boxtimes \mathbf{H}_2}, \mathcal{P}^1_{\mathbf{H}_3 \boxtimes \mathbf{H}_4}) \geq \min_{\nu=3,4}\{d_{\min}(\mathcal{P}^1_{\mathbf{H}_1}, \mathcal{P}^1_{\mathbf{H}_\nu}), d_{\min}(\mathcal{P}^1_{\mathbf{H}_2}, \mathcal{P}^1_{\mathbf{H}_\nu})\}.$$

□

Conclusion and future directions of research

This work defined and studied the Cartesian product as well as the strong product of weighted hypergraphs. Additionally, the paper introduces the notions of clique filtration for the weighted and unweighted hypergraphs as well as for its Cartesian and strong products. Looking into the relation between the bottleneck distance of Cartesian products and the bottleneck distance of individual hypergraphs, the study revealed that the strong product of weighted and unweighted hypergraphs yielded similar results. In future studies, it is recommended to look in to the direct product and the lexicographic product as potential methods to simplify the comparison of large hypergraph networks. Alternative methods for comparing and analyzing complex hypergraph networks, especially those that can be expressed as lexicographic or direct products, may be offered by these product operations.

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